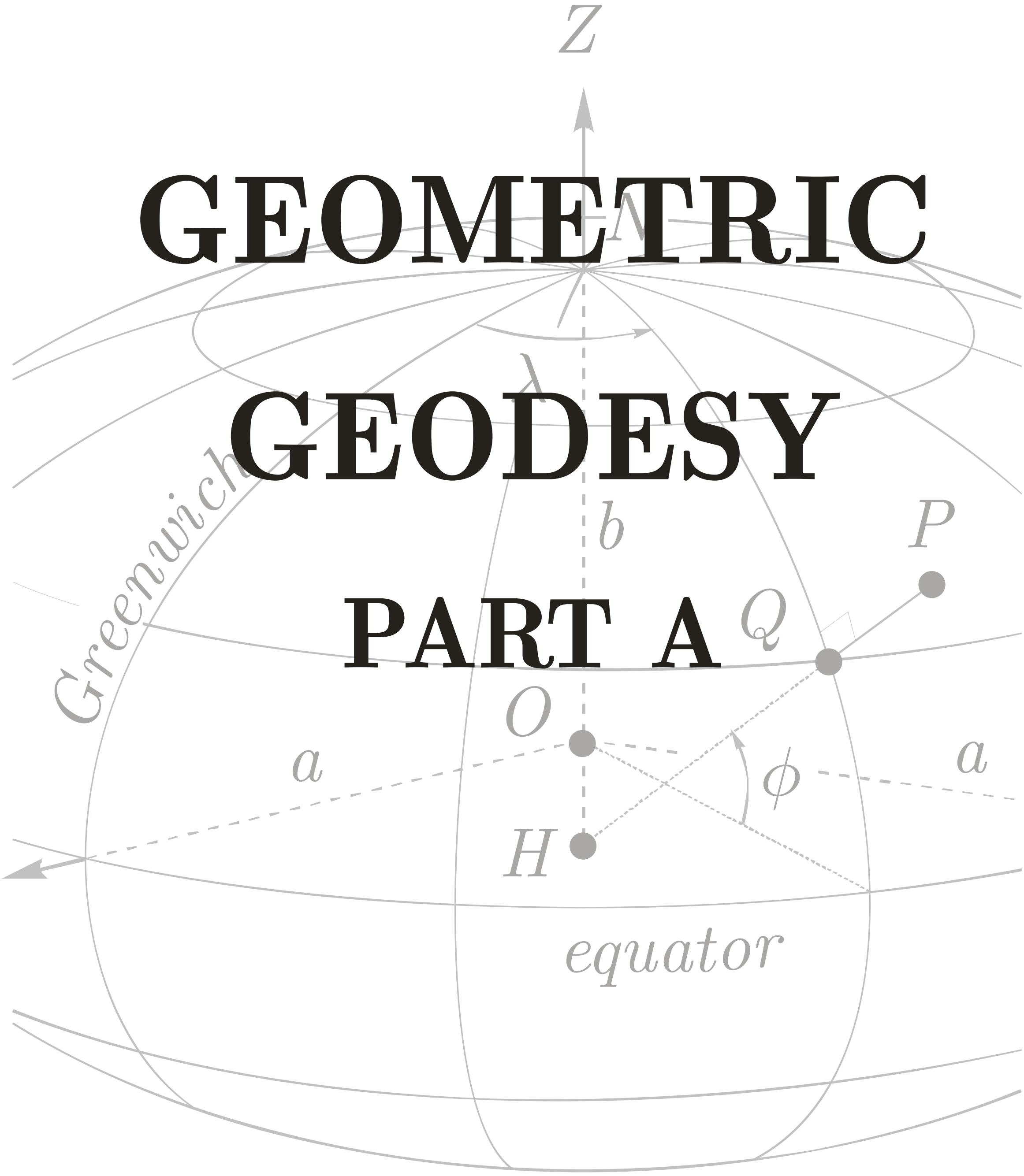
GEOMETRIC GEODESY

PART A



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FOREWORD

These notes are an introduction to ellipsoidal geometry related to geodesy. Many computations in geodesy are concerned with the position of points on the Earth's surface and direction and distance between points. The Earth's surface (the terrestrial surface) is highly irregular and unsuitable for any mathematical computations, instead a reference surface, known as an ellipsoid – a surface of revolution created by rotating an ellipse about its minor axis – is adopted and points on the Earth's terrestrial surface are projected onto the ellipsoid, via a normal to the ellipsoid. All computations are made using these projected points on the ellipsoidal reference surface; hence there is a need to understand the geometry of the ellipsoid.

These notes are intended for undergraduate students studying courses in surveying, geodesy and map projections. The derivations of equations given herein are detailed, and in some cases elementary, but they do convey the vital connection between geodesy and the mathematics taught to undergraduate students.

The information in the notes is drawn from a number of sources; in particular we have followed closely upon the works of G. B. Lauf, *Geodesy and Map Projections* and R. H. Rapp, *Geometric Geodesy,* and also 'Geodesy' a set of notes produced by the New South Wales Department of Technical and Further Education (Tafe).

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1 PROPERTIES OF THE ELLIPSOID

The Earth is a viscous fluid body, rotating in space about its axis that passes through the poles and centre of mass and this axis of revolution is inclined to its orbital plane of rotation about the Sun. The combination of gravitational and rotational forces causes the Earth to be slightly flattened at the poles and the gently undulating equipotential surfaces of the Earth's gravity field also have this characteristic. A particular equipotential surface, the geoid, represents global mean sea level, and since the seas and oceans cover approximately 70% of the Earth's surface, the geoid is a close approximation of the Earth's true shape. The geoid is a gently undulating surface that is difficult to define mathematically, and hence is not a useful reference surface for computation.

A better reference surface is an ellipsoid, which in geodesy is taken to mean a surface of revolution created by rotating an ellipse about its minor axis. Ellipsoids, with particular geometric properties, can be located in certain ways so as to be approximations of the global geoid, or approximations of regional portions of the geoid; this gives rise to geocentric or local reference ellipsoids. In any case, the size and shape of ellipsoids are easily defined mathematically and they are relatively simple surface to compute upon; although not as simple as the sphere. Knowledge of the geometry of the ellipsoid and its generator, the ellipse, is an important part of the study of geodesy.

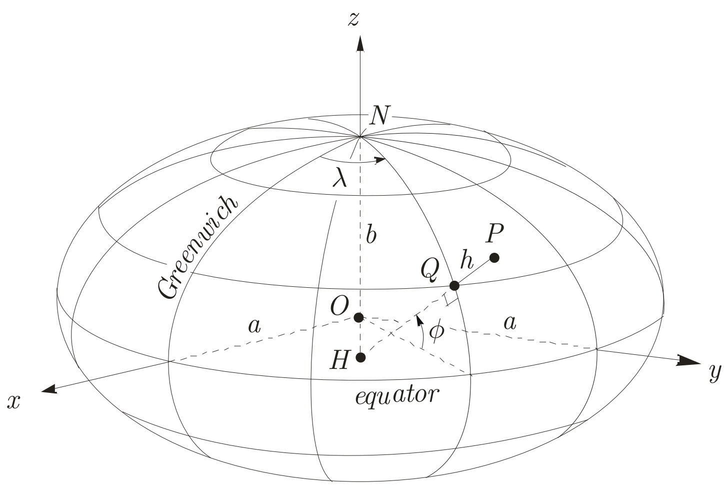


Figure 1: The reference ellipsoid

Figure 1 show a schematic view of the reference ellipsoid upon which meridians (curves of constant longitude λ) and parallels (curves of constant latitude ϕ) form an orthogonal network of reference curves on the surface. This allows a point P in space to be coordinated via a normal to the ellipsoid passing through P. This normal intersects the surface at Q which has coordinates of ϕ,λ and P is at a height $h = QP$ above the ellipsoid surface. We say that P has geodetic coordinates, ϕ,λ,h. P also has Cartesian coordinates $x,y,z;$ but more about these coordinate systems later. The important thing at this stage is that the ellipsoid is a surface of revolution created by rotating an ellipse about its minor axis, where this minor axis is assumed to be either the Earth's rotational axis, or a line in space close to the Earth's rotational axis. Meridians of longitude are curves created by intersecting the ellipsoid with a plane containing the minor axis and these curves are ellipses; as are all curves on the ellipsoid created by intersecting planes. Note here that parallels of latitude (including the equator) are circles; since the intersecting plane is perpendicular to the rotational axis, and circles are just special cases of ellipses. Clearly, an understanding of the ellipse is important in ellipsoidal geometry and thus geometric geodesy.

1.1 THE ELLIPSE

The ellipse is one of the conic sections; a name derived from the way they were first studied, as sections of a cone[1]. A right-circular cone is a solid whose surface is obtained by rotation a straight line, called the generator, about a fixed axis.

In Figure 2, the generator makes an angle γ with the axis and as it is swept around the axis is describes the surface that appears to be two halves of the cone, known as nappes, that touch at a common apex. The generator of a cone in any of its positions is called an element.

[1] The ellipse, parabola and hyperbola, as sections of a cone, were first studied by Menaechmus (circa 380 BC - 320 BC), the Greek mathematician who tutored Alexander the Great. Euclid of Alexandra (circa 325 BC - 265 BC) investigated the ellipse in his treatise on geometry: *The Elements.* Apollonius of Perga (circa 262 BC - 190 BC) in his famous book *Conics* introduced the terms parabola, ellipse and hyperbola and Pappus of Alexandra (circa 290 - 350) introduced the concept of focus and directrix in his studies of projective geometry.

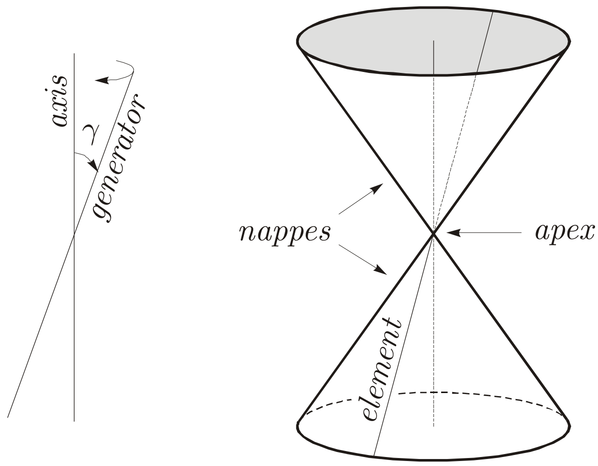


Figure 2: The cone and its generator

The conic sections are the curves created by the intersections of a plane with one or two nappes of the cone.

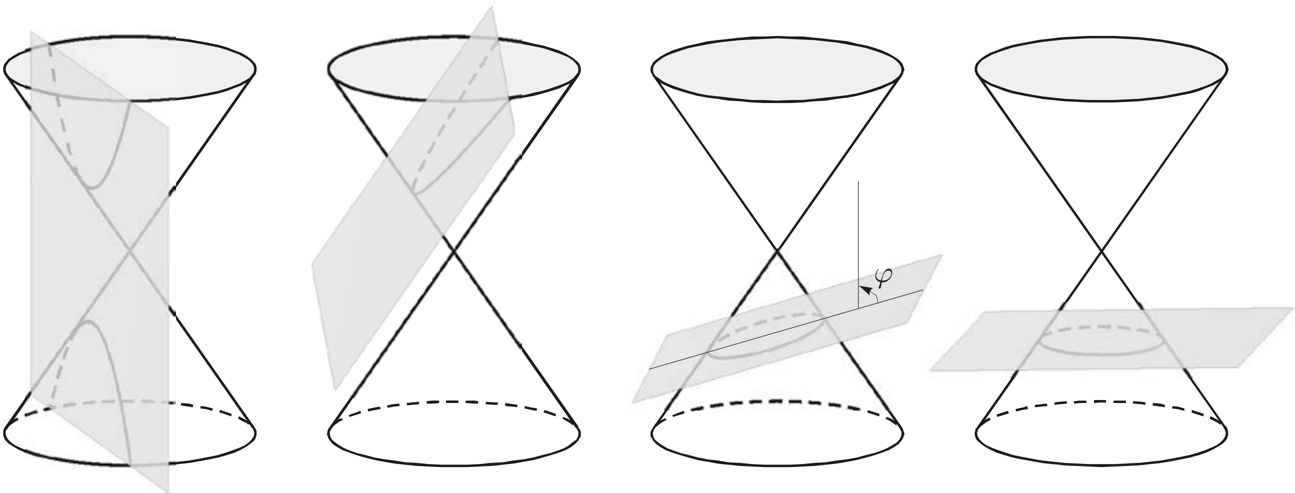


Figure 3: The conic sections

Depending on the angle φ between the axis of the cone and the plane, the conic sections are: *hyperbola* $(0 \le \varphi < \gamma)$, *parabola* $(\varphi = \gamma)$, *ellipse* $\left(\gamma < \varphi < \pi/2\right)$, or *circle* $\left(\varphi = \pi/2\right)$. Note that for $0 \le \varphi < \gamma$ the plane intersects both nappes of the cone and the hyperbola consists of two separate curves.

1.1.1 The equations of the ellipse

An ellipse can be defined in the following three ways:

(a) An ellipse is the locus of a point P_k that moves so that the sum of the distances r and r' from two fixed points F and F' (the foci) separated by a distance $2d$ is a constant and equal to the major axis of the ellipse, i.e.,

$$r + r' = 2a \tag{1}$$

a is the semi-major, b is the semi-minor axis and $d = OF = OF'$ is the focal distance. The origin of the x,y coordinate system is at O, the centre of the ellipse.

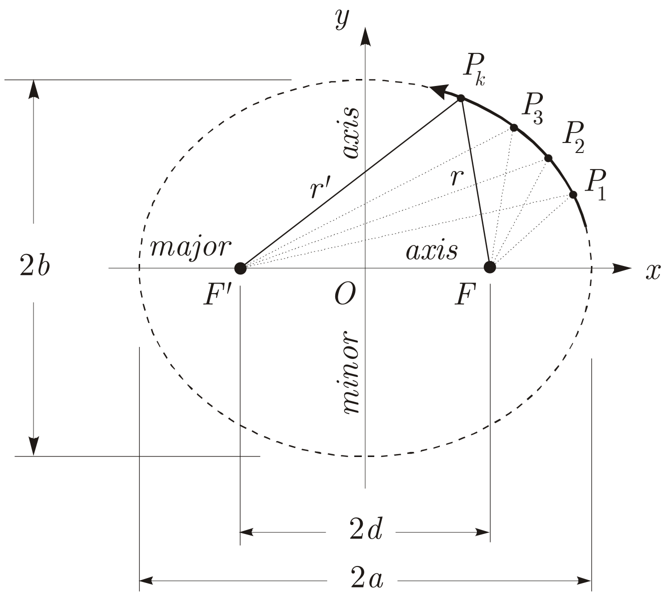


Figure 4: Ellipse

This definition leads to the Cartesian equation of the ellipse. From Figure 4 and equation (1) we may write

$$\sqrt{(x+d)^2 + y^2} + \sqrt{(x-d)^2 + y^2} = 2a$$

Squaring both sides and re-arranging gives

$$\begin{aligned} 4a\sqrt{(x-d)^2 + y^2} &= 4a^2 + (x-d)^2 - (x+d)^2 \\ &= 4a^2 - 4xd \end{aligned}$$

and

$$\sqrt{(x-d)^2 + y^2} = a - \frac{d}{a}x$$

Squaring both sides and gathering the x-terms gives

$$x^2\left(\frac{a^2 - d^2}{a^2}\right) + y^2 = a^2 - d^2 \tag{2}$$

Now, from Figure 4, when P_k on the ellipse is also on the minor axis, $r = r' = a$ and from a right-angled triangle we obtain

$$a^2 = b^2 + d^2; \quad b^2 = a^2 - d^2; \quad d^2 = a^2 - b^2 \tag{3}$$

Substituting the second of equations (3) into equation (2) and simplifying gives the Cartesian equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4}$$

(b) If auxiliary circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ are drawn on a common origin O of an x,y coordinate system and radial lines are drawn at angles ψ from the x-axis; then the ellipse is the locus of points P_k that lie at the intersection of lines, parallel with the coordinate axes, drawn through the intersections of the radial lines and auxiliary circles.

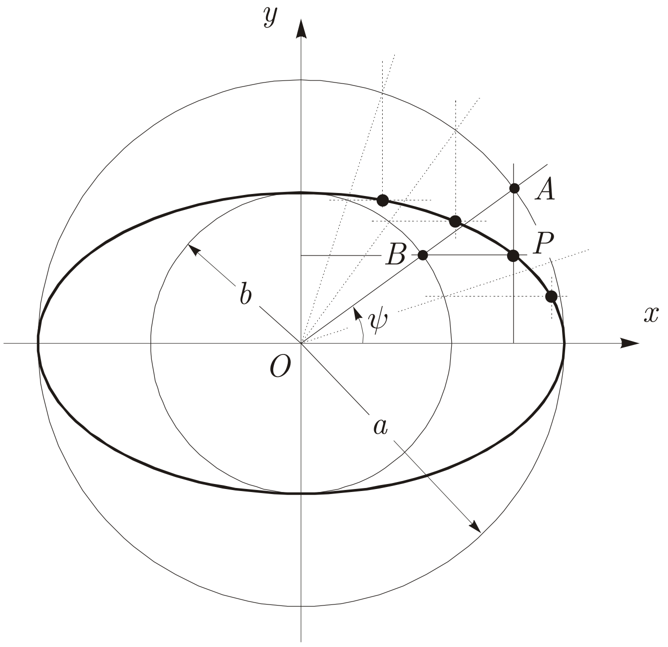


Figure 5: Ellipse and auxiliary circles

This definition leads to the parametric equation of the ellipse. Consider points A (auxiliary circle) and P (ellipse) on Figure 5. Using equation (4) and the equation for the auxiliary circle of radius a we may write

$$x_A^2 + y_A^2 = a^2 \quad \text{and} \quad \frac{x_P^2}{a^2} + \frac{y_P^2}{b^2} = 1$$

and these equations may be re-arranged as

$$x_A^2 = a^2 - y_A^2 \quad \text{and} \quad x_P^2 = a^2 - \frac{a^2}{b^2} y_P^2 \tag{5}$$

Now the x- coordinates of A and P are the same and so the right-hand sides of equations (5) may be equated, giving

$$a^2 - y_A^2 = a^2 - \frac{a^2}{b^2} y_P^2$$

This leads to the relationship

$$y_P = \frac{b}{a} y_A \tag{6}$$

Hence, we may say that the y-coordinate of the ellipse, for an arbitrary x-coordinate, is b/a times the y-coordinate for the circle of radius a at the same value of x.

Now, we can use equation (6) and Figure 5 to write the following equations

$$\begin{aligned} x_A &= a\cos\psi \\ y_A &= a\sin\psi \end{aligned} \quad \text{and} \quad \begin{aligned} x_P &= x_A \\ y_P &= \frac{b}{a} y_A \end{aligned}$$

From which we can write parametric equations for the ellipse

$$\boxed{\begin{aligned} x &= a\cos\psi \\ y &= b\sin\psi \end{aligned}} \tag{7}$$

Similarly, considering points B and P; using equation (4) and the equation for the auxiliary circle of radius b we may write

$$x_B^2 + y_B^2 = b^2 \quad \text{and} \quad \frac{x_P^2}{a^2} + \frac{y_P^2}{b^2} = 1$$

and these equations may be re-arranged as

$$y_B^2 = b^2 - x_B^2 \quad \text{and} \quad y_P^2 = b^2 - \frac{b^2}{a^2} x_P^2 \tag{8}$$

Now the y- coordinates of B and P are the same and so the right-hand sides of equations (8) may be equated, giving

$$b^2 - x_B^2 = b^2 - \frac{b^2}{a^2} x_P^2$$

This leads to the relationship

$$x_P = \frac{a}{b} x_B \tag{9}$$

And we may say that the x-coordinate of the ellipse, for an arbitrary y-coordinate, is a/b times the x-coordinate for the circle of radius b at the same value of y.

Using equation (9) and Figure 5 we have

$$\begin{aligned} x_B &= b\cos\psi \\ y_B &= b\sin\psi \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_P &= \frac{a}{b}x_B \\ y_P &= y_B \end{aligned}$$

giving, as before, equations (7); the parametric equations for an ellipse.

Note that squaring both sides of equations (7) gives $x^2 = a^2\cos^2\psi$ and $y^2 = b^2\sin^2\psi$ and these can be re-arranged as $\frac{x^2}{a^2} = \cos^2\psi$ and $\frac{y^2}{b^2} = \sin^2\psi$. Then using the trigonometric identity $\sin^2\psi + \cos^2\psi = 1$ we obtain the Cartesian equation of the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(c) An ellipse may be defined as the locus of a point P that moves so that its distance from a fixed point F, called the *focus,* bears a constant ratio, that is less than unity, to its distance from a fixed line known as the *directrix,* i.e.,

$$\frac{PF}{PN} = e \tag{10}$$

where e is the *eccentricity* and $e < 1$ for an ellipse.

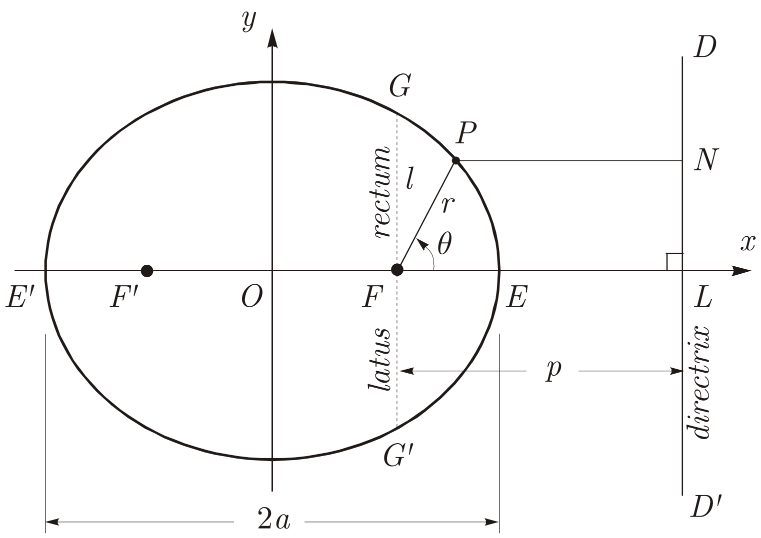


Figure 6: Ellipse (focus-directrix)

From Figure 6 and definition (c), the following relationships may be obtained

$$\frac{FE}{EL} = e \quad \text{and} \quad \frac{FE'}{E'L} = e$$

giving $FE = e(FL)$, $FE' = e(E'L)$ and $FE + FE' = e(EL + E'L)$.

Now since $FE + FE' = 2(OE) = 2a$ and $EL + E'L = 2(OL)$ we may write

$$OL = \frac{a}{e} \tag{11}$$

Also

$$\begin{aligned} FE' - FE &= e(E'L - EL) \\ EE' - 2(FE) &= e(EE') \\ EE'(1-e) &= 2(FE) \\ 2a(1-e) &= 2(FE) \end{aligned}$$

hence

$$FE = a(1-e) \tag{12}$$

And, since $EE' - 2(FE) = 2(OF)$ and $EE' = 2a$ the focal length OF is given by

$$OF = ae \tag{13}$$

In Figure 6, the line GG', perpendicular to the major axis and passing through the focus F is known as the *latus rectum*[2] and $l = FG$ is the *semi latus rectum.*

Using equations (11) and (13), the perpendicular distance from G to the directrix DD' is $OL - OF = \frac{a}{e} - ae$, and employing definition (c) gives $\dfrac{l}{\frac{a}{e} - ae} = e$ and the *semi latus rectum* of the ellipse is

$$l = a(1 - e^2) \tag{14}$$

In Figure 6, $p = FL = OL - OF$ where OL and OF are given by equations (11) and (13).

[2] *Latus rectum* means "side erected" and the length of the latus rectum was used by the ancient Greek mathematicians as a means of defining ellipses, parabolas and hyperbolas.

Using these results gives

$$p = \frac{a}{e}\left(1 - e^2\right) \tag{15}$$

Also, in Figure 6, let $PF = r$, θ be the angle between PF and the x-axis; then $PN = p - r\cos\theta$. Using definition (c), $\frac{PF}{PN} = e$, hence $\frac{r}{e} = p - r\cos\theta$, which can be re-arranged to give a polar equation of an ellipse (with respect to the focus F)

$$r = \frac{ep}{1 + e\cos\theta} \tag{16}$$

Using equations (15) and (14) gives two other results

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta} \tag{17}$$

$$r = \frac{l}{1 + e\cos\theta} \tag{18}$$

Another polar equation of the ellipse can be developed considering Figure 7.

Let $OP = r$ and θ be the angle between OP and the x-axis, then

$$x = r\cos\theta \qquad x^2 = r^2\cos^2\theta$$
$$\text{and}$$
$$y = r\sin\theta \qquad y^2 = r^2\sin^2\theta$$

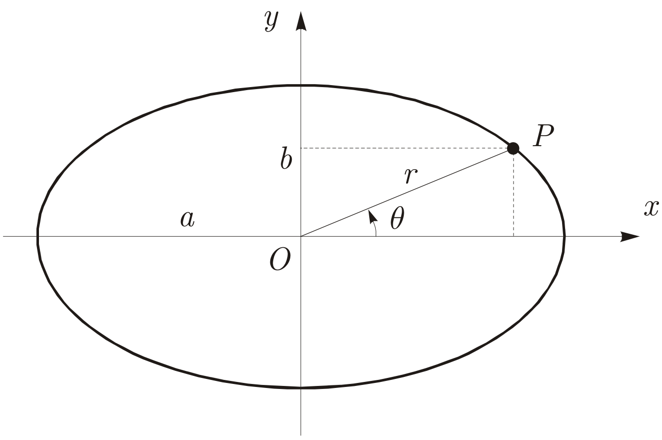


Figure 7: Ellipse (polar equation)

Substituting these expressions for x^2 and y^2 into the Cartesian equation for the ellipse [equation (4)] and re-arranging gives a polar equation of the ellipse (with respect to the origin O)

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \tag{19}$$

or

$$r = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \tag{20}$$

1.1.2 The eccentricities *e* and *e′* of the ellipse

The eccentricity of an ellipse is denoted by e. From Figure 6 and equation (13) it can be defined as

$$e = \frac{OF}{a} \tag{21}$$

From Figure 2 and equations (3) $OF = d = \sqrt{a^2 - b^2}$ and $e = \dfrac{\sqrt{a^2 - b^2}}{a}$. The more familiar way that eccentricity e is defined in geodesy is by its squared-value e^2 as

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} \tag{22}$$

Another eccentricity that is used in geodesy is the 2nd-eccentricity, usually denoted as e' and similarly to the (1st) eccentricity e, the 2nd-eccentricity e' is defined by its squared-value e'^2 as

$$e'^2 = \frac{a^2 - b^2}{b^2} = \frac{a^2}{b^2} - 1 \tag{23}$$

1.1.3 The flattening *f* of the ellipse

The flattening of an ellipse, denoted by f, (and also called the compression or ellipticity) is the ratio which the excess of the semi-major axis over the semi-minor axis bears to the semi-major axis. The flattening f is defined as

$$f = \frac{a - b}{a} = 1 - \frac{b}{a} \tag{24}$$

1.1.4 The ellipse parameters *c, m* and *n*

In certain geodetic formula, the constants c, m and n are used. They are defined as

$$c = \frac{a^2}{b} \tag{25}$$

$$m = \frac{a^2 - b^2}{a^2 + b^2} \tag{26}$$

$$n = \frac{a - b}{a + b} \tag{27}$$

Note: c is the polar radius of the ellipsoid and m is sometimes called the 3rd-eccentricity squared. A 2nd-flattening is defined as $f' = \left(a - b\right)\big/ b$ with a 3rd-flattening as $f'' = n = \left(a - b\right)\big/\left(a + b\right)$.

1.1.5 Interrelationship between ellipse parameters

The ellipse parameters a, b, c, e, e', m and n are related as follows

$$a = b\sqrt{1+e'^2} = \frac{b}{\sqrt{1-e^2}} = c\left(\frac{1-n}{1+n}\right) = c(1-f) \tag{28}$$

$$b = a(1-f) = a\sqrt{1-e^2} = \frac{a}{\sqrt{1+e'^2}} = \frac{c}{1+e'^2} = c\left(1-e^2\right) \tag{29}$$

$$\frac{b}{a} = (1-f) = \sqrt{1-e^2} = \frac{1}{\sqrt{1+e'^2}} = \frac{e}{e'} = \frac{1-n}{1+n} = \sqrt{\frac{1-m}{1+m}} \tag{30}$$

$$f = 1-\sqrt{1-e^2} = 1-\frac{1}{\sqrt{1+e'^2}} = \frac{2n}{1+n} \tag{31}$$

$$e^2 = \frac{e'^2}{1+e'^2} = f(2-f) = \frac{4n}{(1+n)^2} = \frac{2m}{1+m} \tag{32}$$

$$1-e^2 = (1-f)^2 \tag{33}$$

$$e'^2 = \frac{e^2}{1-e^2} = \frac{f(2-f)}{(1-f)^2} = \frac{4n}{(1-n)^2} = \frac{2m}{1-m} \tag{34}$$

$$\left(1-e^2\right)\left(1+e'^2\right) = 1 \tag{35}$$

$$m = \frac{f(2-f)}{1+(1-f)^2} = \frac{2n}{1+n^2} \tag{36}$$

$$n = \frac{f}{2-f} = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} = \frac{\sqrt{1+e'^2}-1}{\sqrt{1+e'^2}+1} \tag{37}$$

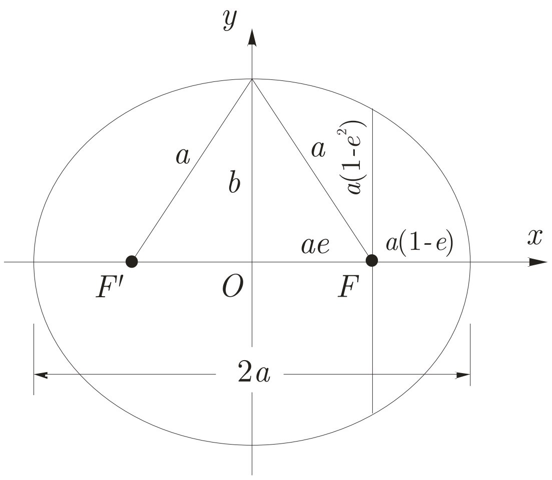


Figure 8: Ellipse geometry

1.1.6 Geometry of the ellipse

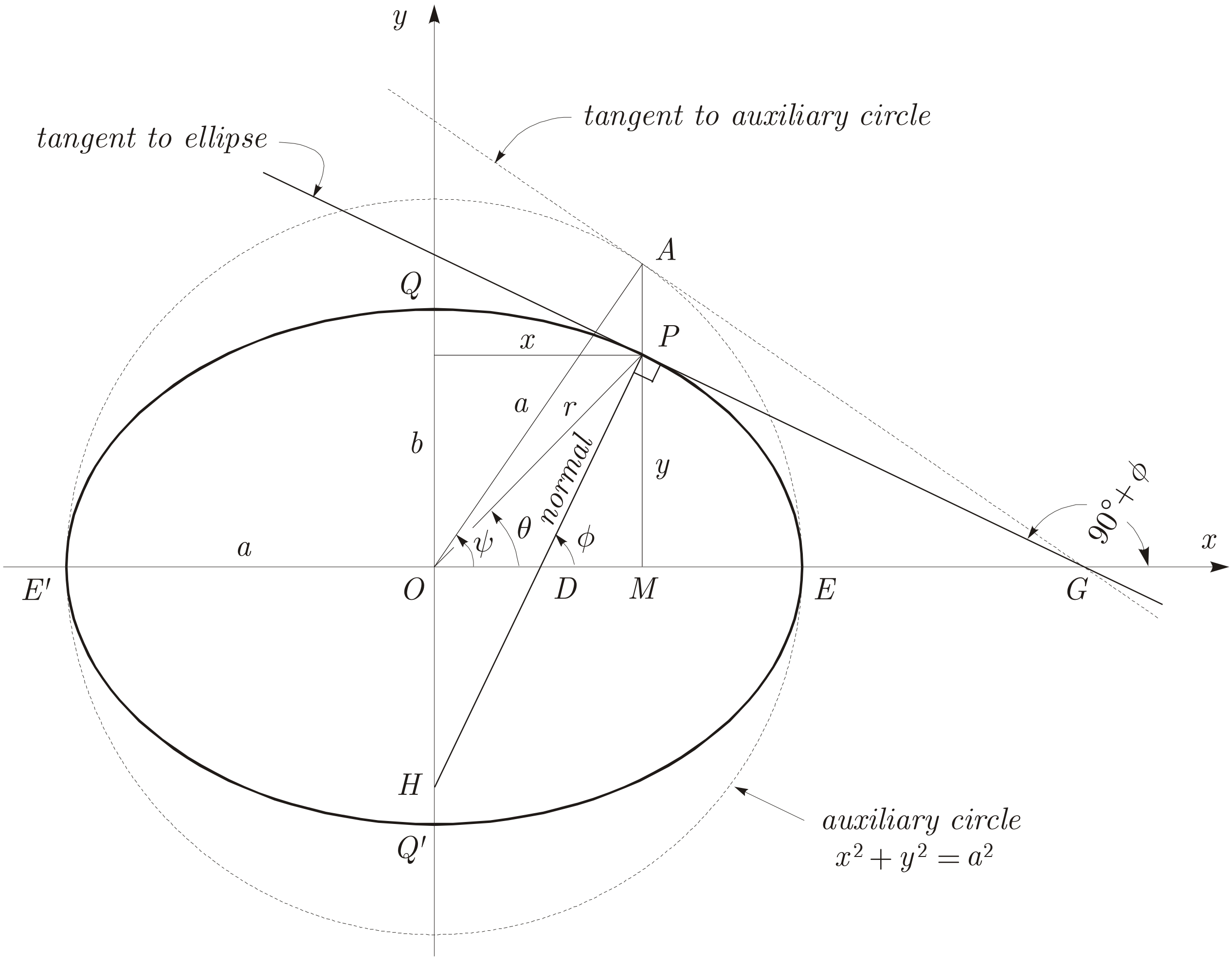


Figure 9: Ellipse and auxiliary circle

In Figure 9, the angles ϕ, ψ and θ are known as *latitudes* and are respectively, angles between the major axis of the ellipse and (i) the normal to the ellipse at P, (ii) a normal to the auxiliary circle at A, and (iii) the radial OP. The x,y Cartesian coordinates of P can be expressed as functions of ϕ and relationships between ϕ, ψ and θ established. These functions can then be used to define distances PH, PD and OH in terms of the ellipse parameters a and e^2. These will be useful in later sections of these notes.

<u>x and y in terms of</u> ϕ

Differentiating equation (4) with respect to x gives

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

and re-arranging gives

$$\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$$

Now by definition, $\frac{dy}{dx}$ is the gradient of the tangent to the ellipse, and from Figure 9

$$\frac{dy}{dx}=\tan\left(90^\circ+\phi\right)=-\cot\phi=-\frac{b^2}{a^2}\frac{x}{y} \tag{38}$$

from which we obtain

$$y=-\frac{b^2}{a^2}x\tan\phi \tag{39}$$

and

$$x=\frac{a^2}{b^2}\frac{y}{\tan\phi} \tag{40}$$

Substituting equation (39) into the Cartesian equation for the ellipse (4) gives

$$\frac{x^2}{a^2}+\frac{b^2}{a^4}x^2\tan^2\phi=1$$

$$\frac{x^2}{a^2}\left\{1+\frac{b^2}{a^2}\frac{\sin^2\phi}{\cos^2\phi}\right\}=1$$

Now, from equation (30) $\frac{b^2}{a^2}=1-e^2$ hence

$$\frac{x^2}{a^2}\left\{1+\left(1-e^2\right)\frac{\sin^2\phi}{\cos^2\phi}\right\}=1$$

$$\frac{x^2}{a^2}\left\{\frac{\cos^2\phi+\sin^2\phi-e^2\sin^2\phi}{\cos^2\phi}\right\}=1$$

$$\frac{x^2}{a^2}\left\{\frac{1-e^2\sin^2\phi}{\cos^2\phi}\right\}=1$$

giving

$$x=\frac{a\cos\phi}{\left(1-e^2\sin^2\phi\right)^{1/2}} \tag{41}$$

Similarly, substituting equation (40) into the Cartesian equation for the ellipse (4) gives

$$\frac{a^2}{b^4}\frac{y^2}{\tan^2\phi}+\frac{y^2}{b^2}=1$$
$$\frac{y^2}{b^2}\left\{1+\frac{a^2}{b^2}\frac{\cos^2\phi}{\sin^2\phi}\right\}=1$$

Now, from equation (30) $\frac{a^2}{b^2}=\frac{1}{1-e^2}$ hence

$$\frac{y^2}{b^2}\left\{1+\frac{\cos^2\phi}{\left(1-e^2\right)\sin^2\phi}\right\}=1$$
$$\frac{y^2}{b^2}\left\{\frac{\sin^2\phi-e^2\sin^2\phi+\cos^2\phi}{\left(1-e^2\right)\sin^2\phi}\right\}=1$$
$$\frac{y^2}{b^2}\left\{\frac{1-e^2\sin^2\phi}{\left(1-e^2\right)\sin^2\phi}\right\}=1$$

giving

$$y=\frac{b\sqrt{1-e^2}\sin\phi}{\left(1-e^2\sin^2\phi\right)^{1/2}} \tag{42}$$

Equations (41) and (42) may be conveniently expressed as another set of parametric equations for the ellipse

$$\begin{aligned} x&=\frac{a}{W}\cos\phi\\ y&=\frac{b\sqrt{1-e^2}}{W}\sin\phi\\ W^2&=1-e^2\sin^2\phi \end{aligned} \tag{43}$$

Equivalent expressions may be obtained for x and y by using the 2nd-eccentricity e'^2. Substituting for e^2 [using equation (32)] in the third of equations (43) gives

$$\begin{aligned} W^2&=1-\frac{e'^2}{1+e'^2}\sin^2\phi\\ &=\frac{1+e'^2-e'^2\sin^2\phi}{1+e'^2}\\ &=\frac{1+e'^2\left(1-\sin^2\phi\right)}{1+e'^2} \end{aligned}$$

and $W^2 = \dfrac{1+e'^2\cos^2\phi}{1+e'^2}$. Putting $V^2 = 1+e'^2\cos^2\phi$ and using equation (30) gives $W^2 = \dfrac{V^2}{1+e'^2} = \dfrac{b^2}{a^2}V^2$. Using these relationships gives another set of parametric equations for the ellipse

$$\begin{aligned} x &= \frac{a^2\cos\phi}{bV} = \frac{c}{V}\cos\phi \\ y &= \frac{b}{V}\sin\phi \\ c &= \frac{a^2}{b} \\ V^2 &= 1+e'^2\cos^2\phi \end{aligned} \tag{44}$$

Also the relationships between W and V may be useful

$$W^2 = 1-e^2\sin^2\phi;\;\; V^2 = 1+e'^2\cos^2\phi \;\text{ and }\; c = \frac{a^2}{b} \tag{45}$$

$$W^2 = \frac{b^2}{a^2}V^2 = \frac{b}{c}V^2 = V^2\left(1-e^2\right) = \frac{V^2}{1+e'^2} \tag{46}$$

$$V^2 = \frac{a^2}{b^2}W^2 = \frac{c}{b}W^2 = \frac{W^2}{1-e^2} = W^2\left(1+e'^2\right) \tag{47}$$

<u>Length of normal terminating on minor axis (*PH*)</u>

$$PH = \frac{x}{\cos\phi} = \frac{a}{\left(1-e^2\sin^2\phi\right)^{1/2}} = \frac{a}{W} = \frac{c}{V} \tag{48}$$

<u>Length of normal terminating on major axis (*PD*)</u>

$$PD = \frac{y}{\sin\phi} = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{1/2}} = PH\left(1-e^2\right) = \frac{a}{W}\left(1-e^2\right) = \frac{c}{V}\left(1-e^2\right) \tag{49}$$

<u>Length *DH* along normal</u>

$$DH = PH - PD = \frac{a}{W}e^2 = \frac{c}{V}e^2 \tag{50}$$

<u>Length *OH* along minor axis</u>

$$OH = DH\sin\phi = \frac{a}{W}e^2\sin\phi = \frac{c}{V}e^2\sin\phi \tag{51}$$

<u>Relationship between latitudes</u>

Differentiating the equations (7) with respect to ψ gives

$$\frac{dx}{d\psi} = -a\sin\psi; \qquad \frac{dy}{d\psi} = b\cos\psi$$

then

$$\frac{dy}{dx} = \frac{dy}{d\psi}\frac{d\psi}{dx} = -\frac{b}{a}\cot\psi \tag{52}$$

Now by definition, $\frac{dy}{dx}$ is the gradient of the tangent to the ellipse, and

$$\frac{dy}{dx} = \tan\left(90^\circ + \phi\right) = -\cot\phi \tag{53}$$

Equating equations (52) and (53) gives relationships between ψ and ϕ

$$\tan\psi = \frac{b}{a}\tan\phi = \sqrt{1-e^2}\,\tan\phi = \left(1-f\right)\tan\phi \tag{54}$$

Also, from Figure 9 and equations (43)

$$\tan\theta = \frac{y}{x} = \frac{a\left(1-e^2\right)\sin\phi}{W}\frac{W}{a\cos\phi}$$

giving relationships between θ and ϕ

$$\tan\theta = \left(1-e^2\right)\tan\phi = \frac{b^2}{a^2}\tan\phi = \left(1-f\right)^2\tan\phi \tag{55}$$

And with equations (54) and (55) the relationships between θ and ψ are

$$\tan\theta = \sqrt{1-e^2}\,\tan\psi = \frac{b}{a}\tan\psi = \left(1-f\right)\tan\psi \tag{56}$$

1.1.7 Curvature

To calculate distances on ellipses (and ellipsoids) we need to know something about the curvature of the ellipse. Curvature at a point on an ellipse can be determined from general relationships applicable to any curve.

The curvature κ (kappa) of a curve $y = y(x)$ at any point P on the curve, is the rate of change of direction of the curve with respect to the arc length; (i.e., the rate of change in the direction of the tangent with respect to the arc length). The curvature is defined as

$$\kappa = \lim_{\delta s \to 0} \frac{\delta\alpha}{\delta s} = \frac{d\alpha}{ds} \tag{57}$$

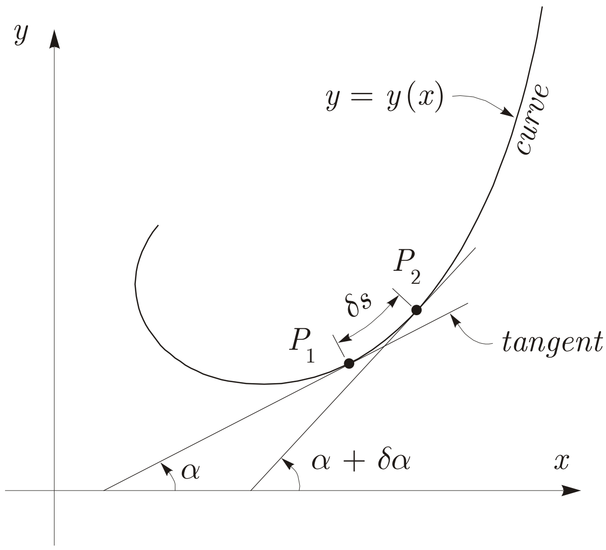
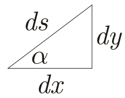


Figure 10: Curvature

The gradient of the tangent to the curve is, by definition, (the 1st-derivative) $\dfrac{dy}{dx} = \tan\alpha$, and the 2nd-derivative is

$$\frac{d^2y}{dx^2} = \sec^2\alpha \frac{d\alpha}{dx} = \sec^2\alpha \frac{d\alpha}{ds}\frac{ds}{dx} \tag{58}$$

But, from equation (57), $\kappa = \dfrac{d\alpha}{ds}$ and from the elemental triangle 

we obtain $\dfrac{ds}{dx} = \dfrac{1}{\cos\alpha} = \sec\alpha$. Substituting these results into equation (58) gives $\dfrac{d^2y}{dx^2} = \kappa\sec^3\alpha$ and a re-arrangement gives the curvature as

$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\sec^3\alpha} \tag{59}$$

The denominator of equation (59) can be simplified by using the trigonometric identity $\sec^2\alpha = 1 + \tan^2\alpha$; so $\sec\alpha = \pm\left(1+\tan^2\alpha\right)^{1/2}$ and $\sec^3\alpha = \pm\left(1+\tan^2\alpha\right)^{3/2}$. Now, since $\dfrac{dy}{dx} = \tan\alpha$, then $\left(\dfrac{dy}{dx}\right)^2 = \tan^2\alpha$, thus $\sec^3\alpha = \pm\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}$. This result for $\sec^3\alpha$ can be substituted into equation (59) to give the equation for curvature as

$$\kappa = \pm \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}} = \pm \frac{y''}{\left\{1+\left(y'\right)^2\right\}^{3/2}} \tag{60}$$

$$\text{where } \frac{dy}{dx} = y' \text{ and } \frac{d^2y}{dx^2} = y''$$

1.1.8 Radius of curvature

The radius of curvature ρ (rho) for a point $P(x,y)$ on a curve $y = y(x)$ is defined as being

$$\rho = \frac{1}{|\kappa|} \text{ for } \kappa \neq 0 \text{ and } \rho = \infty \text{ for } \kappa = 0 \tag{61}$$

The radius of curvature is the radius of the *osculating* (kissing) circle that approximates the curve at that point.

In Figure 11, the radius of curvature at P is $\rho = CP$ and $C(u,v)$ is the centre of curvature whose coordinates are $x = u,\ y = v$.

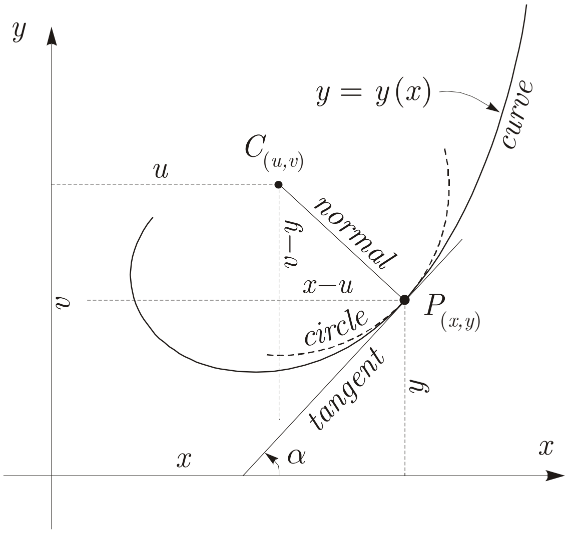


Figure 11: Centre of Curvature

An equation for the radius of curvature ρ can be derived in the following manner.

From equation (38) the gradient of the tangent to the ellipse is

$$y' = \frac{dy}{dx} = -\cot\phi = -\frac{\cos\phi}{\sin\phi} \tag{62}$$

and the 2nd-derivative is

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{d\phi}\left(\frac{-1}{\tan\phi}\right)\frac{d\phi}{dx} = \frac{1}{\sin^2\phi}\frac{d\phi}{dx} \tag{63}$$

The derivative $\dfrac{d\phi}{dx}$ can be obtained from equation (41) where

$$x = \frac{a\cos\phi}{\left(1-e^2\sin^2\phi\right)^{1/2}}$$

and using the quotient rule for differentiation: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$ gives

$$\begin{aligned}\frac{dx}{d\phi}&=\frac{\left(1-e^2\sin^2\phi\right)^{1/2}\left(-a\sin\phi\right)-a\cos\phi\left(\frac{1}{2}\left(1-e^2\sin^2\phi\right)^{1/2}\right)2e^2\sin\phi\cos\phi}{1-e^2\sin^2\phi}\\&=\frac{a\sin\phi}{\left(1-e^2\sin^2\phi\right)^{3/2}}\left\{1-e^2\sin^2\phi-e^2\cos^2\phi\right\}\\&=\frac{a\sin\phi}{\left(1-e^2\sin^2\phi\right)^{3/2}}\left\{1-e^2\left(\sin^2\phi+\cos^2\phi\right)\right\}\\&=\frac{a\left(1-e^2\right)\sin\phi}{\left(1-e^2\sin^2\phi\right)^{3/2}}\end{aligned}$$

hence

$$\frac{d\phi}{dx}=\frac{\left(1-e^2\sin^2\phi\right)^{3/2}}{a\left(1-e^2\right)\sin\phi}\tag{64}$$

Substituting equation (64) into equation (63) gives

$$y''=\frac{\left(1-e^2\sin^2\phi\right)^{3/2}}{a\left(1-e^2\right)\sin^3\phi}\tag{65}$$

Now the equation for curvature (61) can be written as

$$\rho^{2/3}=\frac{1+\left(y'\right)^2}{y''^{2/3}}$$

and substituting equations (62) and (65) gives

$$\begin{aligned}\rho^{2/3}&=\left(1+\frac{\cos^2\phi}{\sin^2\phi}\right)\frac{a^{2/3}\left(1-e^2\right)^{2/3}\sin^2\phi}{1-e^2\sin^2\phi}\\&=\frac{a^{2/3}\left(1-e^2\right)^{2/3}}{1-e^2\sin^2\phi}\end{aligned}$$

giving the equation for radius of curvature ρ for the ellipse as

$$\rho=\frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{3/2}}=\frac{a\left(1-e^2\right)}{W^3}=\frac{c}{V^3}\tag{66}$$

Note that equations (45), (46) and (47) have been used in the simplification.

1.1.9 Centre of curvature

In Figure 11, the centre of curvature for a point $P(x,y)$ on a curve $y = y(x)$ is $C(u,v)$ which is the centre of the osculating circle of radius ρ that approximates the curve at P; and C lies on the normal to the curve at P.

The coordinates $x = u$, $y = v$ of the centre of curvature can be obtained from equation (61) and the general equations of a tangent and a normal to a curve:

$$\begin{aligned} \text{tangent:}\quad & y - y_0 = m(x - x_0) \\ \text{normal:}\quad & y - y_0 = -\frac{1}{m}(x - x_0) \\ & m = \tan\alpha = \frac{dy}{dx} = \dot{y} \end{aligned} \tag{67}$$

The centre of curvature $C(u,v)$ lies (i) on the normal passing through $P(x,y)$ and (ii) at a distance ρ from P measured towards the concave side of the curve $y = y(x)$.

This leads to two equations:

(equation of normal)
$$v - y = -\frac{1}{y'}(u - x) \tag{68}$$

(Pythagoras)
$$(u - x)^2 + (v - y)^2 = \rho^2 = \frac{\left(1 + y'^2\right)^3}{y''^2} \tag{69}$$

Re-arranging equation (68) as $(u - x) = -y'(v - y)$ and substituting into equation (69) gives

$$\begin{aligned} \left(-y'\right)^2 (v - y)^2 + (v - y)^2 &= \frac{\left(1 + y'^2\right)^3}{y''^2} \\ (v - y)^2 \left\{1 + y'^2\right\} &= \frac{\left(1 + y'^2\right)^3}{y''^2} \\ (v - y)^2 &= \frac{\left(1 + y'^2\right)^2}{y''^2} \end{aligned}$$

and

$$v - y = \pm\frac{1 + y'^2}{y''} \tag{70}$$

Note that when the curve is concave upward, $y'' > 0$ and since C lies above P then $v - y > 0$ and the proper sign in equation (70) is +. This is also the case when $y'' < 0$ and the curve is concave downward so

$$v - y = \frac{1 + y'^2}{y''} \tag{71}$$

Substituting equation (71) into equation (68) gives

$$\frac{1 + y'^2}{y''} = -\frac{1}{y'}(u - x) \tag{72}$$

Re-arranging equations (71) and (72) gives the equations for the coordinates (u, v) of the centre of curvature C as

$$\begin{aligned} u &= x - \frac{y'\left(1 + y'^2\right)}{y''} \\ v &= y + \frac{1 + y'^2}{y''} \end{aligned} \tag{73}$$

1.1.10 The evolute of the ellipse

The evolute of a curve is the locus of the centres of curvature.

In Figure 12, the evolute of the ellipse is shown. At P_1 the ellipse has a radius of curvature ρ_1 and the centre of curvature is at C_1, at P_2 the radius of curvature is ρ_2 and centre of curvature at C_2 and at P_3 the radius of curvature is ρ_3 and centre of curvature at C_3. The evolute is the curve joining all the possible centres of curvature.

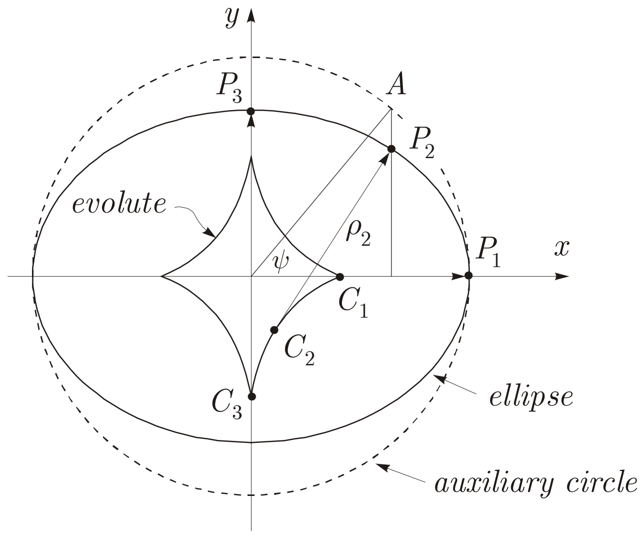


Figure 12: Ellipse, evolute and auxiliary circle

Parametric equations of the evolute are obtained in the following manner.

Parametric equations of the ellipse are given by equations (7) as

$$x = a\cos\psi; \quad y = b\sin\psi$$

Differentiating with respect to ψ gives

$$\frac{dx}{d\psi} = -a\sin\psi; \quad \frac{dy}{d\psi} = b\cos\psi$$

and the chain-rule for differentiation gives the gradient of the tangent to the ellipse as

$$\frac{dy}{dx} = \frac{dy}{d\psi}\frac{d\psi}{dx} = -\frac{b}{a\tan\psi} \quad \text{or} \quad y' = -\frac{b}{a\tan\psi} \tag{74}$$

The second-derivative is

$$\frac{d^2y}{dx^2} = \frac{b}{a\sin^2\psi}\frac{d\psi}{dx} = -\frac{b}{a^2\sin^3\psi} \quad \text{or} \quad y'' = -\frac{b}{a^2\sin^3\psi} \tag{75}$$

Substituting equations (74) and (75) into the equations for the centre of curvature (73) gives

$$u = x - \frac{y'\left(1+y'^2\right)}{y''} = x - \left\{\frac{-\dfrac{b\cos\psi}{a\sin\psi} - \dfrac{b^3\cos^3\psi}{a^3\sin^3\psi}}{-\dfrac{b}{a^2\sin^3\psi}}\right\}$$

expanding the right-hand-side gives

$$\begin{aligned} u &= x - \left\{\frac{-\dfrac{b\cos\psi}{a\sin\psi} - \dfrac{b^3\cos^3\psi}{a^3\sin^3\psi}}{-\dfrac{b}{a^2\sin^3\psi}}\right\} \\ &= x - \left\{\frac{\left(-a^2\sin^2\psi\; b\cos\psi - b^3\cos^3\psi\right)}{a^3\sin^3\psi}\frac{\left(-a^2\sin^3\psi\right)}{b}\right\} \\ &= x - \left\{\frac{a^2\sin^2\psi\cos\psi + b^2\cos^3\psi}{a}\right\} \\ &= \frac{ax - \left\{a^2\cos\psi\left(1-\cos^2\psi\right) + b^2\cos^3\psi\right\}}{a} \end{aligned}$$

and since $x = a\cos\psi$

$$au = a^2\cos\psi - a^2\cos\psi + a^2\cos^3\psi + b^2\cos^3\psi$$

then

$$au = \left(a^2 - b^2\right)\cos^3\psi \tag{76}$$

Similarly, substituting equations (74) and (75) into the equations for the centre of curvature (73) gives

$$v = y + \frac{1 + y'^2}{y''} = y + \left\{\frac{1 + \dfrac{b^2\cos^2\psi}{a^2\sin^2\psi}}{-\dfrac{b}{a^2\sin^3\psi}}\right\}$$

expanding the right-hand-side gives

$$\begin{aligned} v &= y + \left\{\left(1 + \frac{b^2\cos^2\psi}{a^2\sin^2\psi}\right)\left(\frac{-a^2\sin^3\psi}{b}\right)\right\} \\ &= y + \left\{\frac{\left(-a^2\sin^2\psi - b^2\cos^2\psi\right)}{a^2\sin^2\psi}\frac{a^2\sin^3\psi}{b}\right\} \\ &= y - \left\{\frac{a^2\sin^3\psi + b^2\cos^2\psi\sin\psi}{b}\right\} \\ &= \frac{by - \left\{a^2\sin^3\psi + b^2\left(1 - \sin^2\psi\right)\sin\psi\right\}}{b} \end{aligned}$$

and since $y = b\sin\psi$

$$bv = b^2\sin\psi - a^2\sin^3\psi - b^2\sin\psi + b^2\sin^3\psi$$

then

$$bv = -\left(a^2 - b^2\right)\sin^3\psi \tag{77}$$

Using equations (76) and (77); and equations (22) and (23), a set of parametric equations of the evolute of an ellipse are

$$\begin{aligned} x &= \left(\frac{a^2 - b^2}{a}\right)\cos^3\psi &= ae^2\cos^3\psi \\ y &= -\left(\frac{a^2 - b^2}{b}\right)\sin^3\psi &= -be'^2\sin^3\psi \end{aligned} \tag{78}$$

1.2 SOME DIFFERENTIAL GEOMETRY

To establish some properties of the ellipsoid, differential geometry is useful for our purposes; where we take differential geometry to mean the study of curves and surfaces by means of calculus. Using differential geometry we are able to define a geodesic, which is a special curve on an ellipsoid defining the shortest path between two points, and give two theorems; *Meunier's theorem* and *Euler's theorem* that are fundamental to geometric geodesy. These two theorems enable us to derive equations for radii of curvature of normal sections of the ellipsoid and equations for mean radii of curvature. Differential geometry relies heavily on vector representation of curves and surfaces and the two vector products; the *dot* (or scalar) product and the *cross* (or vector) product. Some familiarity with these terms (and manipulations) and vector notation is assumed.

1.2.1 Differential Geometry of Space Curves

A space curve may be defined as the locus of the terminal points P of a position vector $\mathbf{r}(t)$ defined by a single scalar parameter t,

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \tag{79}$$

$\mathbf{i},\mathbf{j},\mathbf{k}$ are fixed unit Cartesian vectors in the directions of the x,y,z coordinate axes. As the parameter t varies the terminal point P of the vector sweeps out the space curve C. Let s be the arc-length of C measured from some convenient point on C, so that

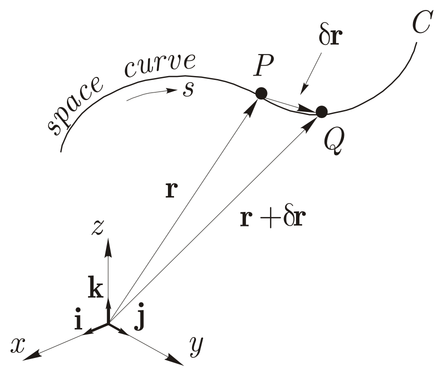


Figure 13: Space curve C

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \quad \text{or} \quad \frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\frac{d\mathbf{r}}{dt}\bullet\frac{d\mathbf{r}}{dt}} \quad \text{and} \quad s = \int \sqrt{\frac{d\mathbf{r}}{dt}\bullet\frac{d\mathbf{r}}{dt}}\, dt$$

Hence s is a function of t and x,y,z are functions of s.

[Note that $\mathbf{a}\bullet\mathbf{b}$ denotes the *dot product* (or scalar product) of two vectors and if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$. $|\mathbf{a}|,|\mathbf{b}|$ are magnitudes or lengths of the vectors, θ is the angle between them and the dot product is a scalar quantity equal to the projection of the length of **a** onto **b**. If **a** is orthogonal to **b**, then $\mathbf{a}\bullet\mathbf{b} = 0$.]

Let Q, a small distance δs along the curve from P, have a position vector $\mathbf{r}+\delta\mathbf{r}$. Then $\delta\mathbf{r}=\overrightarrow{PQ}$ and $|\delta\mathbf{r}|\simeq|\delta s|$. Both when δs is positive or negative $\dfrac{\delta\mathbf{r}}{\delta s}$ approximates to a unit vector in the direction of s increasing and $\dfrac{d\mathbf{r}}{ds}$ is a tangent vector of unit length denoted by $\hat{\mathbf{t}}$; hence

$$\hat{\mathbf{t}}=\frac{d\mathbf{r}}{ds}=\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k} \tag{80}$$

Since $\hat{\mathbf{t}}$ is a unit vector then $\hat{\mathbf{t}}\bullet\hat{\mathbf{t}}=1$ and differentiating with respect to s using the rule $\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$ gives $\dfrac{d}{ds}\left(\hat{\mathbf{t}}\bullet\hat{\mathbf{t}}\right)=\hat{\mathbf{t}}\bullet\dfrac{d\hat{\mathbf{t}}}{ds}+\hat{\mathbf{t}}\bullet\dfrac{d\hat{\mathbf{t}}}{ds}=2\left(\hat{\mathbf{t}}\bullet\dfrac{d\hat{\mathbf{t}}}{ds}\right)=0$. This leads to $\hat{\mathbf{t}}\bullet\dfrac{d\hat{\mathbf{t}}}{ds}=0$ from which we deduce that $\dfrac{d\hat{\mathbf{t}}}{ds}$ is a vector orthogonal to $\hat{\mathbf{t}}$ and write

$$\frac{d\hat{\mathbf{t}}}{ds}=\mathbf{k}=\kappa\hat{\mathbf{n}}, \qquad \kappa>0 \tag{81}$$

$\dfrac{d\hat{\mathbf{t}}}{ds}$ is called the curvature vector $\mathbf{k}$, and should not be confused with the unit vector in the direction of the z-axis. $\hat{\mathbf{n}}$ is a unit vector called the principal normal vector, κ the curvature and $\dfrac{1}{|\kappa|}=\rho$ is the radius of curvature. The circle through P, tangent to $\hat{\mathbf{t}}$ with this radius ρ is called the osculating circle. Also $\hat{\mathbf{n}}\bullet\dfrac{d\hat{\mathbf{t}}}{ds}=\kappa$; i.e., $\hat{\mathbf{n}}$ is the unit vector in the direction of **k**.

Let $\hat{\mathbf{b}}$ be a third unit vector defined by the vector cross product

$$\hat{\mathbf{b}}=\hat{\mathbf{t}}\times\hat{\mathbf{n}} \tag{82}$$

thus $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{b}}$ form a right-handed triad.

[Note that $\mathbf{a}\times\mathbf{b}$ denotes the *cross product* (or vector product) of two vectors and if $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then $\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{p}}=\mathbf{p}$. $|\mathbf{a}|,|\mathbf{b}|$ are magnitudes, θ is the angle between the vectors and $\hat{\mathbf{p}}$ is a unit vector of the vector **p** that is perpendicular to the plane containing **a** and **b**. The direction of **p** is given by the *right-hand-screw rule,* i.e., if **a** and **b** are in the plane of the head of a screw, then a clockwise rotation of **a** to **b** through an angle θ would mean that the direction of **p** would be the same as the direction of advance of a right-handed screw turned clockwise. The cross product can be written as the expansion of a determinant as

$$\mathbf{p} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \overset{(+)}{\mathbf{i}} & \overset{(-)}{\mathbf{j}} & \overset{(+)}{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(a_2 b_3 - a_3 b_2\right)\mathbf{i} - \left(a_1 b_3 - a_3 b_1\right)\mathbf{j} + \left(a_1 b_2 - a_2 b_1\right)\mathbf{k}$$

Note here that the mnemonics $(+)$, $(-)$, $(+)$ are an aid to the evaluation of the determinant. The perpendicular vector $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ has scalar components $p_1 = \left(a_2 b_3 - a_3 b_2\right)$, $p_2 = -\left(a_1 b_3 - a_3 b_1\right)$ and $p_3 = \left(a_1 b_2 - a_2 b_1\right)$. The magnitude (or geometric length) of $\mathbf{p}$ is denoted as $|\mathbf{p}|$ and $|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$ and the unit vector of $\mathbf{p}$, denoted as $\hat{\mathbf{p}}$ is $\hat{\mathbf{p}} = \dfrac{\mathbf{p}}{|\mathbf{p}|} = \dfrac{p_1}{|\mathbf{p}|}\mathbf{i} + \dfrac{p_2}{|\mathbf{p}|}\mathbf{j} + \dfrac{p_3}{|\mathbf{p}|}\mathbf{k}$.]

Differentiating equation (82) with respect to s gives

$$\frac{d\hat{\mathbf{b}}}{ds} = \frac{d}{ds}\left(\hat{\mathbf{t}} \times \hat{\mathbf{n}}\right) = \frac{d\hat{\mathbf{t}}}{ds} \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds} = \kappa\hat{\mathbf{n}} \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds} = \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}$$

then

$$\hat{\mathbf{t}} \bullet \frac{d\hat{\mathbf{b}}}{ds} = \hat{\mathbf{t}} \bullet \left(\hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}\right) = \frac{d\hat{\mathbf{n}}}{ds} \bullet \left(\hat{\mathbf{t}} \times \hat{\mathbf{t}}\right) = 0$$

so that $\dfrac{d\hat{\mathbf{b}}}{ds}$ is orthogonal to $\hat{\mathbf{t}}$. But from $\hat{\mathbf{b}} \bullet \hat{\mathbf{b}} = 1$ it follows that $\hat{\mathbf{b}} \bullet \dfrac{d\hat{\mathbf{b}}}{ds} = 0$ so that $\dfrac{d\hat{\mathbf{b}}}{ds}$ is orthogonal to $\hat{\mathbf{b}}$ and so is in the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$.

Since $\dfrac{d\hat{\mathbf{b}}}{ds}$ is in the plane of $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$, and is orthogonal to $\hat{\mathbf{t}}$, it must be parallel to $\hat{\mathbf{n}}$. The direction of $\dfrac{d\hat{\mathbf{b}}}{ds}$ is opposite $\hat{\mathbf{n}}$ as it must be to ensure the cross product $\dfrac{d\hat{\mathbf{b}}}{ds} \times \hat{\mathbf{t}}$ is in the direction of $\hat{\mathbf{b}}$. Hence

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau\hat{\mathbf{n}}, \qquad \tau > 0 \qquad (83)$$

We call $\hat{\mathbf{b}}$ the unit <u>binormal vector</u>, τ the <u>torsion</u>, and $\dfrac{1}{\tau}$ the <u>radius of torsion</u>. $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ form a right-handed set of orthogonal unit vectors along a space curve.

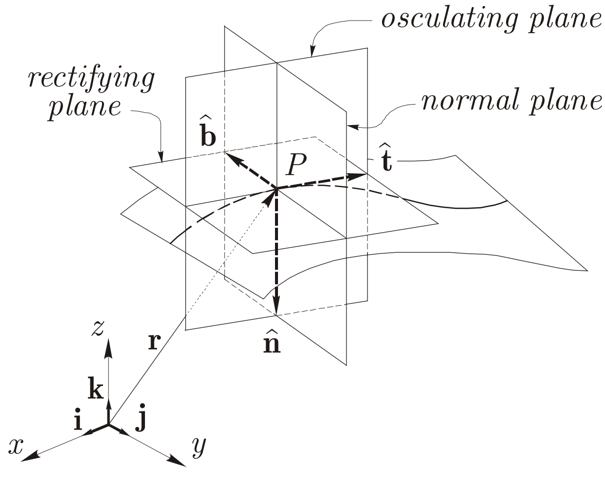


Figure 14: The tangent $\hat{\mathbf{t}}$, principal normal $\hat{\mathbf{n}}$ and binormal $\hat{\mathbf{b}}$ to a space curve

The plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ is the osculating plane, the plane containing $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ is the normal plane and the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{b}}$ is the rectifying plane. Figure 14 shows these orthogonal unit vectors for a space curve.

Also $\hat{\mathbf{n}} = \hat{\mathbf{b}} \times \hat{\mathbf{t}}$ and the derivative with respect to s is

$$\frac{d\hat{\mathbf{n}}}{ds} = \frac{d}{ds}\left(\hat{\mathbf{b}} \times \hat{\mathbf{t}}\right) = \frac{d\hat{\mathbf{b}}}{ds} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \frac{d\hat{\mathbf{t}}}{ds} = -\tau \hat{\mathbf{n}} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \kappa \hat{\mathbf{n}} = \tau \hat{\mathbf{b}} - \kappa \hat{\mathbf{t}} \tag{84}$$

Equations (81), (83) and (84) are known as the Frenet-Serret formulae.

$$\begin{aligned} \frac{d\hat{\mathbf{t}}}{ds} &= \kappa \hat{\mathbf{n}} \\ \frac{d\hat{\mathbf{b}}}{ds} &= -\tau \hat{\mathbf{n}} \\ \frac{d\hat{\mathbf{n}}}{ds} &= \tau \hat{\mathbf{b}} - \kappa \hat{\mathbf{t}} \end{aligned} \tag{85}$$

or in matrix notation

$$\begin{bmatrix} d\hat{\mathbf{t}}/ds \\ d\hat{\mathbf{b}}/ds \\ d\hat{\mathbf{n}}/ds \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & -\tau \\ -\kappa & \tau & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{t}} \\ \hat{\mathbf{b}} \\ \hat{\mathbf{n}} \end{bmatrix} \tag{86}$$

These formulae, derived independently by the French mathematicians Jean-Frédéric Frenet (1816–1900) and Joseph Alfred Serret (1819–1885) describe the dynamics of a point moving along a continuous and differentiable curve in three-dimensional space. Frenet derived these formulae in his doctoral thesis at the University of Toulouse; the latter part of which was published as 'Sur quelques propriétés des courbes à double courbure', (some properties of curves with double curvature) in the *Journal de mathématiques pures et appliqués* (Journal of pure and applied mathematics), Vol. 17, pp.437-447, 1852. Frenet also explained their use in a paper titled 'Théorèmes sur les courbes gauches' (Theorems on awkward curves) published in 1853. Serret presented an independent derivation of the same formulae in 'Sur quelques formules relatives à la théorie des courbes à double courbure' (Some formulas relating to the theory of curves with double curvature) published in the *J. de Math.* Vol. 16, pp.241-254, 1851 (DSB 1971).

1.2.2 Radius of curvature of ellipse using differential geometry

As an application of the differential geometry of a space curve, consider the ellipse in the x-y plane in Figure 15. An expression for the curvature κ, and hence the radius of curvature $\rho = 1/\kappa$, can be derived in the following manner.

Using the cross product and the first of the Frenet-Serret formula [equation (85)]

$$\hat{\mathbf{t}} \times \kappa\hat{\mathbf{n}} = \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{t}}}{ds} = \hat{\mathbf{t}} \times \frac{d}{ds}\left(\frac{d\mathbf{r}}{ds}\right) = \hat{\mathbf{t}} \times \frac{d^2\mathbf{r}}{ds^2} \tag{87}$$

Now, from equation (82), $\hat{\mathbf{t}} \times \hat{\mathbf{n}} = \hat{\mathbf{b}}$, so $\hat{\mathbf{t}} \times \kappa\hat{\mathbf{n}} = \kappa\hat{\mathbf{b}}$ and also, from equation (80), $\hat{\mathbf{t}} = \dfrac{d\mathbf{r}}{ds}$; so equation (87) becomes

$$\kappa\hat{\mathbf{b}} = \frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2} \tag{88}$$

Now, since $\hat{\mathbf{b}}$ is a unit vector, then $\left|\kappa\hat{\mathbf{b}}\right| = \kappa\left|\hat{\mathbf{b}}\right| = \kappa$; so taking the magnitude of both sides of equation (88) gives an expression for the curvature as

$$\kappa = \left|\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right| \tag{89}$$

$\mathbf{r}$ is the position vector of P on the ellipse, and $\mathbf{r}$ is given by equation (79) with parametric latitude ψ replacing the general parameter t,

$$\mathbf{r}(\psi) = x(\psi)\mathbf{i} + y(\psi)\mathbf{j} + z(\psi)\mathbf{k} \tag{90}$$

$\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ are the unit tangent vector and unit normal vector respectively, both of which are shown on Figure 15. Note that $\hat{\mathbf{t}}$ is in the direction of increasing parametric latitude ψ and $\hat{\mathbf{n}}$ is directed towards the centre of curvature C.

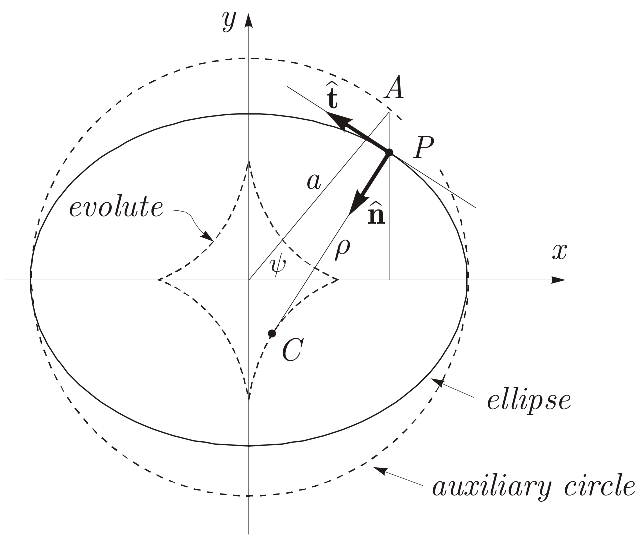


Figure 15:

Using the chain rule for derivatives and the rule $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$, the elements of the right-hand-side of equation (89) can be expressed in terms of the parametric latitude ψ as

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\psi}\frac{d\psi}{ds} \tag{91}$$

And

$$\begin{aligned}
\frac{d^2\mathbf{r}}{ds^2} &= \frac{d}{ds}\left(\frac{d\mathbf{r}}{ds}\right) = \frac{d}{ds}\left(\frac{d\mathbf{r}}{d\psi}\frac{d\psi}{ds}\right) \\
&= \frac{d\mathbf{r}}{d\psi}\frac{d}{ds}\left(\frac{d\psi}{ds}\right) + \frac{d\psi}{ds}\frac{d}{ds}\left(\frac{d\mathbf{r}}{ds}\right) \\
&= \frac{d\mathbf{r}}{d\psi}\frac{d^2\psi}{ds^2} + \frac{d\psi}{ds}\frac{d}{d\psi}\left(\frac{d\mathbf{r}}{d\psi}\right)\frac{d\psi}{ds} \\
&= \frac{d\mathbf{r}}{d\psi}\frac{d^2\psi}{ds^2} + \frac{d^2\mathbf{r}}{d\psi^2}\left(\frac{d\psi}{ds}\right)^2
\end{aligned} \tag{92}$$

Now, substituting equations (91) and (92) into equation (89) gives

$$\begin{aligned}
\kappa &= \left|\left\{\frac{d\mathbf{r}}{d\psi}\frac{d\psi}{ds}\right\}\times\left\{\frac{d\mathbf{r}}{d\psi}\frac{d^2\psi}{ds^2} + \frac{d^2\mathbf{r}}{d\psi^2}\left(\frac{d\psi}{ds}\right)^2\right\}\right| \\
&= \left|\left\{\frac{d\mathbf{r}}{d\psi}\frac{d\psi}{ds}\times\frac{d\mathbf{r}}{d\psi}\frac{d^2\psi}{ds^2}\right\} + \left\{\frac{d\mathbf{r}}{d\psi}\frac{d\psi}{ds}\times\frac{d^2\mathbf{r}}{d\psi^2}\left(\frac{d\psi}{ds}\right)^2\right\}\right| \\
&= \left|0 + \left\{\frac{d\mathbf{r}}{d\psi}\times\frac{d^2\mathbf{r}}{d\psi^2}\right\}\left(\frac{d\psi}{ds}\right)^3\right| \\
&= \left|\frac{d\mathbf{r}}{d\psi}\times\frac{d^2\mathbf{r}}{d\psi^2}\right|\left(\frac{d\psi}{ds}\right)^3
\end{aligned} \tag{93}$$

In equation (93), an expression for the term $\dfrac{d\psi}{ds}$ can be determined as follows. From equations (80) and (90) we may write

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\psi}\frac{d\psi}{ds} = \left(\frac{dx}{d\psi}\mathbf{i} + \frac{dy}{d\psi}\mathbf{j} + \frac{dz}{d\psi}\mathbf{k}\right)\frac{d\psi}{ds}$$

Taking the dot product of the unit vector $\hat{\mathbf{t}}$ with itself gives

$$\hat{\mathbf{t}}\bullet\hat{\mathbf{t}} = 1 = \left\{\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2 + \left(\frac{dz}{d\psi}\right)^2\right\}\left(\frac{d\psi}{ds}\right)^2$$

and we may write

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2 + \left(\frac{dz}{d\psi}\right)^2}} = \frac{1}{\left|\frac{d\mathbf{r}}{d\psi}\right|} \tag{94}$$

Substituting equation (94) into equation (93) gives the expression for curvature as

$$\kappa = \frac{\left| \dfrac{d\mathbf{r}}{d\psi} \times \dfrac{d^2\mathbf{r}}{d\psi^2} \right|}{\left| \dfrac{d\mathbf{r}}{d\psi} \right|^3} \tag{95}$$

We can now use equation (95) to derive an equation for radius of curvature $\rho = \dfrac{1}{|\kappa|}$.

Parametric equations of the ellipse in the x-y plane are [see equations (7)]

$$\begin{aligned} x &= x(\psi) = a\cos\psi \\ y &= y(\psi) = b\sin\psi \\ z &= z(\psi) = 0 \qquad\qquad \text{where } a > 0 \text{ and } b > 0 \end{aligned}$$

and the position vector $\mathbf{r}$ is

$$\mathbf{r} = a\cos\psi\,\mathbf{i} + b\sin\psi\,\mathbf{j} + 0\,\mathbf{k}$$

The derivatives are

$$\begin{aligned} \frac{d\mathbf{r}}{d\psi} &= -a\sin\psi\,\mathbf{i} + b\cos\psi\,\mathbf{j} + 0\,\mathbf{k} \\ \frac{d^2\mathbf{r}}{d\psi^2} &= -a\cos\psi\,\mathbf{i} - b\sin\psi\,\mathbf{j} + 0\,\mathbf{k} \end{aligned}$$

and the cross product in equation (95) is

$$\frac{d\mathbf{r}}{d\psi} \times \frac{d^2\mathbf{r}}{d\psi^2} = \begin{vmatrix} \overset{(+)}{\mathbf{i}} & \overset{(-)}{\mathbf{j}} & \overset{(+)}{\mathbf{k}} \\ -a\sin\psi & b\cos\psi & 0 \\ -a\cos\psi & -b\sin\psi & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + \left(ab\sin^2\psi + ab\cos^2\psi\right)\mathbf{k}$$

and

$$\begin{aligned} \left| \frac{d\mathbf{r}}{d\psi} \times \frac{d^2\mathbf{r}}{d\psi^2} \right| &= \sqrt{0^2 + 0^2 + \left(ab\sin^2\psi + ab\cos^2\psi\right)^2} = ab \\ \left| \frac{d\mathbf{r}}{d\psi} \right|^3 &= \left(a^2\sin^2\psi + b^2\cos^2\psi\right)^{3/2} \end{aligned}$$

Substituting these results into equation (95) and taking the reciprocal gives

$$\rho = \frac{\left(a^2\sin^2\psi + b^2\cos^2\psi\right)^{3/2}}{ab} \tag{96}$$

The term $a^2 \sin^2 \psi + b^2 \cos^2 \psi$ in equation (96) can be simplified in the following manner

$$\begin{aligned} a^2 \sin^2 \psi + b^2 \cos^2 \psi &= \frac{a^2 \sin^2 \psi}{\cos^2 \psi} \cos^2 \psi + b^2 \cos^2 \psi \\ &= \cos^2 \psi \left(a^2 \tan^2 \psi + b^2 \right) \end{aligned} \tag{97}$$

Using equation (54) that gives the relationships been $\tan \psi$ and $\tan \phi$ we may write

$$a^2 \tan^2 \psi = b^2 \tan^2 \phi \tag{98}$$

and from the parametric equations of an ellipse (7) and equations (43) we equate the x-coordinate, which leads to

$$\cos^2 \psi = \frac{\cos^2 \phi}{1 - e^2 \sin^2 \phi} \tag{99}$$

Substituting equations (98) and (99) into equation (97) gives

$$\begin{aligned} \cos^2 \psi \left(a^2 \tan^2 \psi + b^2 \right) &= \frac{\cos^2 \phi}{1 - e^2 \sin^2 \phi} \left(b^2 \tan^2 \phi + b^2 \right) \\ &= \frac{\cos^2 \phi}{1 - e^2 \sin^2 \phi} \left(b^2 \left(1 + \tan^2 \phi \right) \right) \\ &= \frac{\cos^2 \phi}{1 - e^2 \sin^2 \phi} \left(b^2 \sec^2 \phi \right) \\ &= \frac{b^2}{1 - e^2 \sin^2 \phi} \end{aligned} \tag{100}$$

Substituting equation (100) into equation (97) gives

$$a^2 \sin^2 \psi + b^2 \cos^2 \psi = \frac{b^2}{1 - e^2 \sin^2 \phi} \tag{101}$$

Substituting equation (101) into equation (96) gives

$$\rho = \frac{\left(\dfrac{b^2}{1 - e^2 \sin^2 \phi} \right)^{3/2}}{ab} = \frac{b^2}{a^2} \frac{a}{\left(1 - e^2 \sin^2 \phi \right)^{3/2}}$$

and using equations (30) (45) and (47)

$$\rho = \frac{a \left(1 - e^2 \right)}{\left(1 - e^2 \sin^2 \phi \right)^{3/2}} = \frac{a \left(1 - e^2 \right)}{W^3} = \frac{c}{V^3} \tag{102}$$

This is identical to equation (66) which was derived from classical methods.

1.2.3 Differential Geometry of Surfaces

Suppose a surface S is defined by the two-parameter vector equation

$$\mathbf{r} = \mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k} \tag{103}$$

where u and v are independent variables usually called <u>curvilinear coordinates</u>. By holding one of the parameters u or v fixed, the position vector $\mathbf{r}$ traces out parametric curves $u =$ constant and $v =$ constant on the surface S. These parametric curves are also sometimes referred to as u-curves and v-curves.

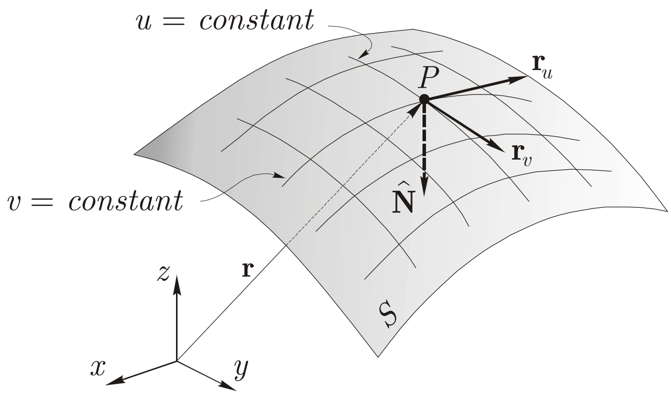


Figure 16: Curved surface with parametric curves u and v

The vectors

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \\ \mathbf{r}_v &= \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} \end{aligned} \tag{104}$$

are both tangent vectors to the surface S and $\mathbf{r}_u$ is tangential to the parametric curve $v =$ constant and $\mathbf{r}_v$ is tangential to the parametric curve $u =$ constant. $\mathbf{r}_u$ and $\mathbf{r}_v$ are not unit vectors and they do not coincide in direction (except perhaps at an isolated point) so that in general $\mathbf{r}_u \times \mathbf{r}_v$ is not a null vector. Higher order derivatives are expressed as

$$\mathbf{r}_{uu} = \frac{\partial}{\partial u}\left(\frac{\partial \mathbf{r}}{\partial u}\right) = \frac{\partial^2 \mathbf{r}}{\partial u^2}, \quad \mathbf{r}_{vv} = \frac{\partial}{\partial v}\left(\frac{\partial \mathbf{r}}{\partial v}\right) = \frac{\partial^2 \mathbf{r}}{\partial v^2}, \quad \mathbf{r}_{uv} = \frac{\partial}{\partial v}\left(\frac{\partial \mathbf{r}}{\partial u}\right) = \frac{\partial^2 \mathbf{r}}{\partial u \partial v}, \quad \text{etc} \tag{105}$$

Using the *Theorem of the Total Differential* (Sokolnikoff & Redheffer 1966) we may write

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv = \mathbf{r}_u\,du + \mathbf{r}_v\,dv \tag{106}$$

and $d\mathbf{r}$ is a position vector known as the <u>first order surface differential</u>.

The <u>second order surface differential</u> $d^2\mathbf{r}$ is given as

$$\begin{aligned} d^2\mathbf{r} &= \frac{\partial}{\partial u}\left\{\frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv\right\}du + \frac{\partial}{\partial v}\left\{\frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv\right\}dv \\ &= \mathbf{r}_{uu}(du)^2 + 2\mathbf{r}_{uv}\,du\,dv + \mathbf{r}_{vv}(dv)^2 \end{aligned} \tag{107}$$

The <u>First Fundamental Form (FFF)</u> of a surface is given by

$$\begin{aligned} (ds)^2 = \text{FFF} &= d\mathbf{r}\bullet d\mathbf{r} \\ &= \left(\mathbf{r}_u\,du + \mathbf{r}_v\,dv\right)\bullet\left(\mathbf{r}_u\,du + \mathbf{r}_v\,dv\right) \\ &= \mathbf{r}_u\bullet\mathbf{r}_u(du)^2 + 2\mathbf{r}_u\bullet\mathbf{r}_v\,du\,dv + \mathbf{r}_v\bullet\mathbf{r}_v(dv)^2 \\ &= E(du)^2 + 2F\,du\,dv + G(dv)^2 \end{aligned} \tag{108}$$

where

$$\begin{aligned} E &= \mathbf{r}_u\bullet\mathbf{r}_u = \left|\mathbf{r}_u\right|^2 \\ F &= \mathbf{r}_u\bullet\mathbf{r}_v \\ G &= \mathbf{r}_v\bullet\mathbf{r}_v = \left|\mathbf{r}_v\right|^2 \end{aligned} \tag{109}$$

are the <u>First Fundamental Coefficients (FFC)</u>.

If $u = u(t), v = v(t)$ are scalar functions of a single scalar parameter t, then

$$\mathbf{r} = \mathbf{r}(u,v) = \mathbf{r}\left(u(t), v(t)\right) \equiv \mathbf{r}(t) \tag{110}$$

is the one-parameter position vector equation of a curve on the surface. The <u>arc-length</u> s of this curve between $t = t_1$ and $t = t_2$ is given by

$$\begin{aligned} s &= \int_{t_1}^{t_2}\left|\frac{d\mathbf{r}}{dt}\right|dt = \int_{t_1}^{t_2}\left|\mathbf{r}_u\frac{du}{dt} + \mathbf{r}_v\frac{dv}{dt}\right|dt = \int_{t_1}^{t_2}\left(\frac{d\mathbf{r}}{dt}\bullet\frac{d\mathbf{r}}{dt}\right)^{1/2}dt \\ &= \int_{t_1}^{t_2}\left[\left(\mathbf{r}_u\frac{du}{dt} + \mathbf{r}_v\frac{dv}{dt}\right)\bullet\left(\mathbf{r}_u\frac{du}{dt} + \mathbf{r}_v\frac{dv}{dt}\right)\right]^{1/2}dt \\ &= \int_{t_1}^{t_2}\left[E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2\right]^{1/2}dt \end{aligned} \tag{111}$$

Also

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \left[E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2\right]^{1/2} \tag{112}$$

Since $\mathbf{r}_u$ and $\mathbf{r}_v$ are tangent vectors along the $v =$ constant and $u =$ constant parametric curves on the surface, then a unit surface normal $\hat{\mathbf{N}}$ is given by

$$\hat{\mathbf{N}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\left|\mathbf{r}_u \times \mathbf{r}_v\right|} \tag{113}$$

with normal vector differential

$$d\hat{\mathbf{N}} = \frac{\partial \hat{\mathbf{N}}}{\partial u} du + \frac{\partial \hat{\mathbf{N}}}{\partial v} dv = \hat{\mathbf{N}}_u \, du + \hat{\mathbf{N}}_v \, dv \tag{114}$$

Note that $d\hat{\mathbf{N}}$ is orthogonal to $\hat{\mathbf{N}}$. This can be proved by the following: (i) $\hat{\mathbf{N}} \bullet \hat{\mathbf{N}} = 1$ and the differential $d\left(\hat{\mathbf{N}} \bullet \hat{\mathbf{N}}\right) = d(1) = 0$; (ii) $d\left(\hat{\mathbf{N}} \bullet \hat{\mathbf{N}}\right) = \hat{\mathbf{N}} \bullet d\hat{\mathbf{N}} + d\hat{\mathbf{N}} \bullet \hat{\mathbf{N}} = 2d\hat{\mathbf{N}} \bullet \hat{\mathbf{N}}$ which leads to (iii) $d\left(\hat{\mathbf{N}} \bullet \hat{\mathbf{N}}\right) = 2d\hat{\mathbf{N}} \bullet \hat{\mathbf{N}} = 0$ giving $d\hat{\mathbf{N}} \bullet \hat{\mathbf{N}} = 0$ and $d\hat{\mathbf{N}}$ is orthogonal to $\hat{\mathbf{N}}$.

An expression for the denominator of equation (113) can be developed using a formula for vector dot and cross products: $(\mathbf{a} \times \mathbf{b}) \bullet (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \bullet \mathbf{c})(\mathbf{b} \bullet \mathbf{d}) - (\mathbf{a} \bullet \mathbf{d})(\mathbf{b} \bullet \mathbf{c})$ giving

$$\begin{aligned} \left|\mathbf{r}_u \times \mathbf{r}_v\right|^2 &= \left(\mathbf{r}_u \times \mathbf{r}_v\right) \bullet \left(\mathbf{r}_u \times \mathbf{r}_v\right) \\ &= \left(\mathbf{r}_u \bullet \mathbf{r}_u\right) \bullet \left(\mathbf{r}_v \bullet \mathbf{r}_v\right) - \left(\mathbf{r}_u \bullet \mathbf{r}_v\right)^2 \\ &= EG - F^2 \end{aligned} \tag{115}$$

Defining a quantity J, that is a function of the First Fundamental Coefficients, as

$$J \equiv \left|\mathbf{r}_u \times \mathbf{r}_v\right| = \sqrt{EG - F^2} \tag{116}$$

we may express the unit surface normal $\hat{\mathbf{N}}$ as

$$\hat{\mathbf{N}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{J} \tag{117}$$

The tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ at P on the surface S, intersect at an angle θ, hence

$$\left|\mathbf{r}_u \times \mathbf{r}_v\right| = \left|\mathbf{r}_u\right|\left|\mathbf{r}_v\right| \sin\theta = \sqrt{EG} \sin\theta \tag{118}$$

and from equations (109)

$$F = \mathbf{r}_u \bullet \mathbf{r}_v = \left|\mathbf{r}_u\right|\left|\mathbf{r}_v\right| \cos\theta = \sqrt{EG} \cos\theta \tag{119}$$

so that the angle between the tangent vectors to the parametric curves on the surface is given by

$$\cos\theta = \frac{F}{\sqrt{EG}} \qquad \text{and} \qquad \sin\theta = \frac{J}{\sqrt{EG}} \tag{120}$$

We can see from this equation that if F is zero, then the parametric curves on the surface S intersect at right angles, i.e., the parametric curves form an orthogonal network on the surface.

If we consider an infinitesimally small quadrilateral on the surface S whose sides are bounded by the curves $u = \text{const.}$, $v = \text{const.}$, $u + du = \text{const.}$ and $v + dv = \text{const.}$ then the lengths of adjacent sides are ds_u and ds_v. These infinitesimal lengths are found from equation (108) by setting $dv = 0$ and $du = 0$ respectively, giving

$$ds_u = \sqrt{E}\,du \quad \text{and} \quad ds_v = \sqrt{G}\,dv \tag{121}$$

This infinitesimally small quadrilateral can be considered as a plane parallelogram whose area is

$$\begin{aligned} dA &= ds_u ds_v \sin\theta \\ &= \sqrt{EG}\sin\theta\,du\,dv \\ &= J\,du\,dv \end{aligned} \tag{122}$$

The <u>Second Fundamental Form (SFF)</u> of a surface is given by

$$\begin{aligned} \text{SFF} = -d\mathbf{r}\bullet d\hat{\mathbf{N}} &= -\left(\mathbf{r}_u\,du + \mathbf{r}_v\,dv\right)\bullet\left(\hat{\mathbf{N}}_u\,du + \hat{\mathbf{N}}_v\,dv\right) \\ &= -\mathbf{r}_u\bullet\hat{\mathbf{N}}_u\,(du)^2 - \left(\mathbf{r}_u\bullet\hat{\mathbf{N}}_v + \mathbf{r}_v\bullet\hat{\mathbf{N}}_u\right)du\,dv + \mathbf{r}_v\bullet\hat{\mathbf{N}}_v\,(dv)^2 \\ &= L\,(du)^2 + 2M\,du\,dv + N\,(dv)^2 \end{aligned} \tag{123}$$

where

$$\begin{aligned} L &= -\mathbf{r}_u\bullet\hat{\mathbf{N}}_u \\ 2M &= -\left(\mathbf{r}_u\bullet\hat{\mathbf{N}}_v + \mathbf{r}_v\bullet\hat{\mathbf{N}}_u\right) \\ N &= -\mathbf{r}_v\bullet\hat{\mathbf{N}}_v \end{aligned} \tag{124}$$

are the <u>Second Fundamental Coefficients (SFC)</u>.

Alternative expressions for the Second Fundamental Form and the Second Fundamental Coefficients can be obtained by the following.

Since $\mathbf{r}_u\bullet\hat{\mathbf{N}} = 0$ and $\mathbf{r}_v\bullet\hat{\mathbf{N}} = 0$ (from the definition of $\hat{\mathbf{N}}$), then

(i) $\dfrac{\partial}{\partial u}\left(\mathbf{r}_u \bullet \hat{\mathbf{N}}\right) = \left(\mathbf{r}_u \bullet \hat{\mathbf{N}}\right)_u = 0$; i.e., $\mathbf{r}_u \bullet \hat{\mathbf{N}}_u + \mathbf{r}_{uu} \bullet \hat{\mathbf{N}} = 0$ and so $\mathbf{r}_u \bullet \hat{\mathbf{N}}_u = -\mathbf{r}_{uu} \bullet \hat{\mathbf{N}}$.

(ii) $\dfrac{\partial}{\partial v}\left(\mathbf{r}_u \bullet \hat{\mathbf{N}}\right) = \left(\mathbf{r}_u \bullet \hat{\mathbf{N}}\right)_v = 0$; i.e., $\mathbf{r}_u \bullet \hat{\mathbf{N}}_v + \mathbf{r}_{uv} \bullet \hat{\mathbf{N}} = 0$ and so $\mathbf{r}_u \bullet \hat{\mathbf{N}}_v = -\mathbf{r}_{uv} \bullet \hat{\mathbf{N}}$.

(iii) $\dfrac{\partial}{\partial u}\left(\mathbf{r}_v \bullet \hat{\mathbf{N}}\right) = \left(\mathbf{r}_v \bullet \hat{\mathbf{N}}\right)_u = 0$; i.e., $\mathbf{r}_v \bullet \hat{\mathbf{N}}_u + \mathbf{r}_{vu} \bullet \hat{\mathbf{N}} = 0$ and so $\mathbf{r}_v \bullet \hat{\mathbf{N}}_u = -\mathbf{r}_{vu} \bullet \hat{\mathbf{N}}$.

(iv) $\dfrac{\partial}{\partial v}\left(\mathbf{r}_v \bullet \hat{\mathbf{N}}\right) = \left(\mathbf{r}_v \bullet \hat{\mathbf{N}}\right)_v = 0$; i.e., $\mathbf{r}_v \bullet \hat{\mathbf{N}}_v + \mathbf{r}_{vv} \bullet \hat{\mathbf{N}} = 0$ and so $\mathbf{r}_v \bullet \hat{\mathbf{N}}_v = -\mathbf{r}_{vv} \bullet \hat{\mathbf{N}}$.

Hence

$$\begin{aligned} L &= \hat{\mathbf{N}} \bullet \mathbf{r}_{uu} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{J} \bullet \mathbf{r}_{uu} \\ M &= \hat{\mathbf{N}} \bullet \mathbf{r}_{uv} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{J} \bullet \mathbf{r}_{uv} \\ N &= \hat{\mathbf{N}} \bullet \mathbf{r}_{vv} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{J} \bullet \mathbf{r}_{vv} \end{aligned} \tag{125}$$

and the Second Fundamental Form (SFF) becomes

$$\begin{aligned} \text{SFF} &= \mathbf{r}_{uu} \bullet \hat{\mathbf{N}}\,(du)^2 + 2\mathbf{r}_{uv} \bullet \hat{\mathbf{N}}\,du\,dv + \mathbf{r}_{vv} \bullet \hat{\mathbf{N}}\,(dv)^2 \\ &= \left[\mathbf{r}_{uu}\,(du)^2 + 2\mathbf{r}_{uv}\,du\,dv + \mathbf{r}_{vv}\,(dv)^2\right] \bullet \hat{\mathbf{N}} \\ &= d^2\mathbf{r} \bullet \hat{\mathbf{N}} \end{aligned} \tag{126}$$

where $d^2\mathbf{r}$ is the <u>second order surface differential</u>, and

$$d^2\mathbf{r} = \mathbf{r}_{uu}\,(du)^2 + 2\mathbf{r}_{uv}\,du\,dv + \mathbf{r}_{vv}\,(dv)^2 \tag{127}$$

Let P be a point on a surface with coordinates (u,v) and Q a neighbouring point on the surface with coordinates $\left(u+du, v+dv\right)$. Using Taylor's theorem, the position vector $\mathbf{r}(u,v)$ can be written as

$$\begin{aligned} \mathbf{r}(u,v) = \mathbf{r}(u_P, v_P) &+ (u-u_P)\,\mathbf{r}_u + (v-v_P)\,\mathbf{r}_v \\ &+ \frac{1}{2!}\left\{(u-u_P)^2\,\mathbf{r}_{uu} + 2(u-u_P)(v-v_P)\,\mathbf{r}_{uv} + (v-v_P)^2\,\mathbf{r}_{vv}\right\} \\ &+ \text{higher order terms} \end{aligned} \tag{128}$$

where all partial derivatives are evaluated u_P, v_P.

Letting $u = u_P + du$ and $v = v_P + dv$, then $\mathbf{r}(u,v) = \mathbf{r}\big(u + du,\ v + dv\big)$, $u - u_P = du$ and $v - v_P = dv$. Substituting into equation (128) gives

$$\begin{aligned}\mathbf{r}\left(u_P + du, v_P + dv\right) &= \mathbf{r}\left(u_P, v_P\right) + \mathbf{r}_u\,du + \mathbf{r}_v\,dv \\ &\quad + \frac{1}{2}\left\{\mathbf{r}_{uu}\left(du\right)^2 + 2\mathbf{r}_{uv}\,du\,dv + \mathbf{r}_{vv}\left(dv\right)^2\right\} \\ &\quad + \text{higher order terms}\end{aligned} \tag{129}$$

Dropping the subscript *P,* then using equation (127) and re-arranging gives

$$\mathbf{r}(u + du, v + dv) - \mathbf{r}(u,v) = d\mathbf{r} + \frac{1}{2}d^2\mathbf{r} + \text{higher order terms} \tag{130}$$

Now $\overrightarrow{PQ} = \mathbf{r}(u + du, v + dv) - \mathbf{r}(u,v)$ and $\overrightarrow{PQ}\bullet\hat{\mathbf{N}}$ is the projection of $\overrightarrow{PQ}$ onto the unit surface normal, so using equation (130) we may write

$$\overrightarrow{PQ}\bullet\hat{\mathbf{N}} = d\mathbf{r}\bullet\hat{\mathbf{N}} + \frac{1}{2}d^2\mathbf{r}\bullet\hat{\mathbf{N}} + \text{higher order terms} \tag{131}$$

Now, using equation (126) and noting that $d\mathbf{r}\bullet\hat{\mathbf{N}} = 0$ (since $d\mathbf{r}$ and $\hat{\mathbf{N}}$ are orthogonal) equation (131) becomes

$$\overrightarrow{PQ}\bullet\hat{\mathbf{N}} = \frac{1}{2}\text{SFF} + \text{higher order terms} \tag{132}$$

This shows that the Second Fundamental Form (SFF) is the principal part of twice the projection of $\overrightarrow{PQ}$ onto $\hat{\mathbf{N}}$ so that $|\text{SFF}|$ is the principal part of twice the perpendicular distance from Q onto the tangent plane to the surface at P. It should be noted here that as $\overrightarrow{PQ} \to 0$ the higher order terms $\to 0$.

In Figure 17, C is a curve on a surface S and P is a point on the curve. $\hat{\mathbf{t}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{n}}$ are the orthogonal unit vectors of the curve C at P and the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ is the osculating plane. Also at *P,* $\hat{\mathbf{N}}$ is the unit normal vector to the surface and the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{N}}$ is the normal section plane. In general, the osculating plane and the normal section plane, both containing the common tangent $\hat{\mathbf{t}}$, do not coincide, but instead make an angle ξ with each other.

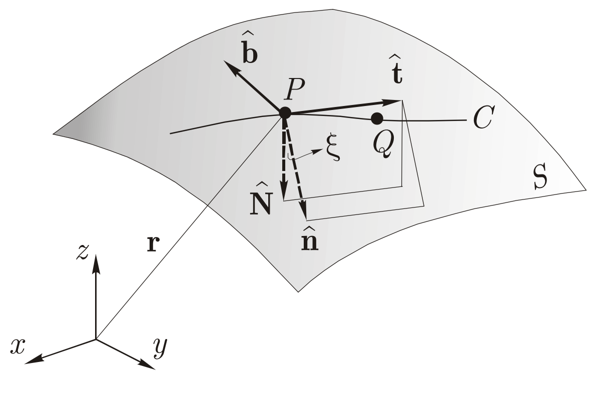


Figure 17: Curve C on surface S

At P on the curve C, the normal curvature vector $\mathbf{k}_N$ is the projection of the curvature vector $\mathbf{k}$ of C onto the surface unit normal $\hat{\mathbf{N}}$, so that $\mathbf{k}_N = \left(\mathbf{k} \bullet \hat{\mathbf{N}}\right)\hat{\mathbf{N}}$. The scalar component κ_N of $\mathbf{k}_N$ in the direction of $\hat{\mathbf{N}}$ is given by

$$\kappa_N = \mathbf{k} \bullet \hat{\mathbf{N}} \tag{133}$$

and κ_N is called the (scalar) normal curvature.

Also, at P on the curve C, the osculating plane containing $\hat{\mathbf{n}}$ and the normal section plane containing $\hat{\mathbf{N}}$ make an angle ξ with each other, hence

$$\left|\hat{\mathbf{n}} \times \hat{\mathbf{N}}\right| = \sin\xi \tag{134}$$

Using equation (133) and the first of the Frenet-Serret formulae (85)

$$\kappa_N = \mathbf{k} \bullet \hat{\mathbf{N}} = \frac{d\hat{\mathbf{t}}}{ds} \bullet \hat{\mathbf{N}} = \kappa \hat{\mathbf{n}} \bullet \hat{\mathbf{N}} = \kappa \cos\xi \tag{135}$$

This is *Meusnier's theorem*[3] that relates the normal curvature κ_N with the curvature κ of a curve on a surface. When $\xi = 0$, $\hat{\mathbf{n}} \times \hat{\mathbf{N}} = \mathbf{0}$; i.e., $\hat{\mathbf{n}}$ and $\hat{\mathbf{N}}$ are (by convention) parallel and pointing in the same direction, and $\kappa_N = \kappa$.

Since $\rho = \dfrac{1}{\kappa}$, Meunier's theorem can also be stated as:

> Between the radius ρ of the osculating circle of a plane section at P and the radius ρ_N of the osculating circle of a normal section at P, where both sections have a common tangent, there exists the relation

$$\boxed{\rho = \rho_N \cos\xi} \tag{136}$$

[3] Meusnier's theorem is a fundamental theorem on the nature of surfaces, named in honour of the French mathematician Jean-Baptiste-Marie-Charles Meusnier de la Place (1754 - 1793) who, in a paper titled *Mémoire sur la corbure des surfaces* (Memoir on the curvature of surfaces), read at the Paris Academy of Sciences in 1776 and published in 1785, derived his theorem on the curvature, at a point on a surface, of plane sections with a common tangent (DSB 1971).

The normal curvature $\kappa_N = \mathbf{k} \bullet \hat{\mathbf{N}}$ is the ratio of the Second Fundamental Form (SFF) and the First Fundamental Form (FFF), or $\kappa_N = \dfrac{\text{SFF}}{\text{FFF}}$. This can be demonstrated by the following.

The unit tangent vector $\hat{\mathbf{t}}$ and the surface unit normal vector $\hat{\mathbf{N}}$ are orthogonal, so $\hat{\mathbf{t}} \bullet \hat{\mathbf{N}} = 0$, and $\dfrac{d}{dt}\left(\hat{\mathbf{t}} \bullet \hat{\mathbf{N}}\right) = 0$. That is $\dfrac{d\hat{\mathbf{t}}}{dt} \bullet \hat{\mathbf{N}} + \hat{\mathbf{t}} \bullet \dfrac{d\hat{\mathbf{N}}}{dt} = 0$, hence $\dfrac{d\hat{\mathbf{t}}}{dt} \bullet \hat{\mathbf{N}} = -\hat{\mathbf{t}} \bullet \dfrac{d\hat{\mathbf{N}}}{dt}$.

Now, using equation (135) we may write

$$\kappa_N = \mathbf{k} \bullet \hat{\mathbf{N}} = \frac{d\hat{\mathbf{t}}}{ds} \bullet \hat{\mathbf{N}}$$

So, by the chain rule

$$\kappa_N = \frac{d\hat{\mathbf{t}}}{ds} \bullet \hat{\mathbf{N}} = \frac{d\hat{\mathbf{t}}}{dt}\frac{dt}{ds} \bullet \hat{\mathbf{N}} = \frac{d\hat{\mathbf{t}}/dt}{ds/dt} \bullet \hat{\mathbf{N}} = -\frac{\mathbf{t} \bullet d\hat{\mathbf{N}}/dt}{ds/dt} = -\frac{d\mathbf{r}/ds \bullet d\hat{\mathbf{N}}/dt}{ds/dt} = -\frac{d\mathbf{r}/dt \bullet d\hat{\mathbf{N}}/dt}{\left(ds/dt\right)^2}$$

Now, using equations (106) and (114) gives

$$\kappa_N = -\frac{\left(\mathbf{r}_u \dfrac{du}{dt} + \mathbf{r}_v \dfrac{dv}{dt}\right) \bullet \left(\mathbf{N}_u \dfrac{du}{dt} + \mathbf{N}_v \dfrac{dv}{dt}\right)}{\left(\dfrac{ds}{dt}\right)^2}$$

and the numerator and denominator can be simplified using equations (123) and (112) respectively and finally the normal curvature κ_N becomes

$$\kappa_N = \mathbf{k} \bullet \hat{\mathbf{N}} = \frac{L\left(\dfrac{du}{dt}\right)^2 + 2M\dfrac{du}{dt}\dfrac{dv}{dt} + N\left(\dfrac{dv}{dt}\right)^2}{E\left(\dfrac{du}{dt}\right)^2 + 2F\dfrac{du}{dt}\dfrac{dv}{dt} + G\left(\dfrac{dv}{dt}\right)^2} = \frac{L\left(du\right)^2 + 2Mdudv + N\left(dv\right)^2}{E\left(du\right)^2 + 2Fdudv + G\left(dv\right)^2} = \frac{\text{SFF}}{\text{FFF}} \tag{137}$$

Dividing the FFF and SFF by $(du)^2$ and making the substitution

$$\lambda = \frac{dv}{du} \tag{138}$$

gives the normal curvature as

$$\kappa_N = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} \tag{139}$$

Note here that λ is an unspecified parametric direction on the surface S.

Extreme values of κ_N can be found by solving $\dfrac{d\kappa_N}{d\lambda} = 0$, that is, using the quotient rule for differentiation $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ gives

$$\frac{d\kappa_N}{d\lambda} = \frac{\left(E + 2F\lambda + G\lambda^2\right)(2M + 2N\lambda) - \left(L + 2M\lambda + N\lambda^2\right)(2F + 2G\lambda)}{\left(E + 2F\lambda + G\lambda^2\right)^2} = 0$$

that simplifies to

$$\left(E + 2F\lambda + G\lambda^2\right)(M + N\lambda) - \left(L + 2M\lambda + N\lambda^2\right)(F + G\lambda) = 0 \tag{140}$$

Now since $E + 2F\lambda + G\lambda^2 = E + F\lambda + \lambda(F + G\lambda)$ and

$L + 2M\lambda + N\lambda^2 = L + M\lambda + \lambda(M + N\lambda)$

equation (140) can be simplified as

$$\begin{aligned} E + F\lambda + \lambda(F + G\lambda) &= L + M\lambda + \lambda(M + N\lambda) \\ \frac{E + F\lambda}{F + G\lambda} + \lambda &= \frac{L + M\lambda}{M + N\lambda} + \lambda \\ \frac{E + F\lambda}{F + G\lambda} &= \frac{L + M\lambda}{M + N\lambda} \end{aligned}$$

and extreme κ_N satisfies

$$\kappa_N = \frac{E + F\lambda}{F + G\lambda} = \frac{L + M\lambda}{M + N\lambda} \tag{141}$$

or

$$\begin{aligned} \kappa_N (F + G\lambda) - (M + N\lambda) &= 0 \\ \kappa_N (E + F\lambda) - (L + M\lambda) &= 0 \end{aligned}$$

that can be re-cast as

$$\begin{aligned} \kappa_N F - M + \left(\kappa_N G - N\right)\lambda &= 0 \\ \kappa_N E - L + \left(\kappa_N F - M\right)\lambda &= 0 \end{aligned} \tag{142}$$

From equation (141) we may write

$$(M + N\lambda)(E + F\lambda) - (L + M\lambda)(F + G\lambda) = 0$$

that can be expressed as a quadratic equation in λ

$$(FN - GM)\lambda^2 + (EN - GL)\lambda + EM - FL = 0 \tag{143}$$

Two values of λ are found, unless SFF vanishes or is proportional to FFF. These values of λ, or $\dfrac{dv}{du}$ are called the directions of principal curvature labelled λ_1 and λ_2 and the normal curvatures in these directions are called the principal curvatures, and labelled κ_1 and κ_2. These principal curvatures are the extreme values of the normal curvature κ_N and correspond with the two values of λ found from equation (143).

The solutions for the quadratic equation $ax^2 + bx + c = 0$ are $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and x_1 and x_2 are real and unequal if a, b, c are real, and $b^2 - 4ac > 0$. Also, $x_1 + x_2 = -b/a$ and $x_1 x_2 = c/a$. Using these relationships we have from equation (143)

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \frac{-(EN - GL) \pm \sqrt{(EN - GL)^2 - 4(FN - GM)(EM - FL)}}{2(FN - GM)} \tag{144}$$

and the sum and product of the parametric directions are

$$\lambda_1 + \lambda_2 = -\frac{EN - GL}{FN - GM} \tag{145}$$

$$\lambda_1 \lambda_2 = -\frac{EM - FL}{FN - GM} \tag{146}$$

Equations (142) can be expressed in the matrix form $\mathbf{Ax} = \mathbf{0}$ as

$$\begin{bmatrix} \kappa_N F - M & \kappa_N G - N \\ \kappa_N E - L & \kappa_N F - M \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These homogeneous equations have non-trivial solutions for $\mathbf{x}$ if, and only if, the determinant of the coefficient matrix $\mathbf{A}$ is zero (Sokolnikoff & Redheffer 1966). This leads to a quadratic equation in κ_N

$$\begin{vmatrix} \kappa_N F - M & \kappa_N G - N \\ \kappa_N E - L & \kappa_N F - M \end{vmatrix} = \left(EG - F^2\right)\kappa_N^2 - (EN + GL - 2FM)\kappa_N + LN - M^2 = 0$$

whose solutions are $\kappa_N = \kappa_1$ and $\kappa_N = \kappa_2$, the principal curvatures. Half the sum of the solutions and the product of the solutions can be used to define two other curvatures:

(i) average curvature $$\tfrac{1}{2}\left(\kappa_1 + \kappa_2\right) = \frac{EN + GL - 2FM}{2\left(EG - F^2\right)} = \frac{EN + GL - 2FM}{2J^2} \tag{147}$$

(ii) Gaussian curvature $$\kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} = \frac{LN - M^2}{J^2} \tag{148}$$

We will now show that the principal directions are orthogonal. Consider two curves C_1 and C_2 on a surface S with curvilinear coordinates u, v. The infinitesimal distance ds along C_1 corresponds to infinitesimal changes du and dv along the parametric curves $u =$ constant and $v =$ constant. Similarly, an infinitesimal distance δs along C_2 corresponds to infinitesimal changes δu and δv. Furthermore, the two curves are in the directions of the principal curvatures k_1 and κ_2, and these principal directions are defined as $\lambda_1 = \dfrac{dv}{du}$ and $\lambda_2 = \dfrac{\delta v}{\delta u}$.

Using equation (106) we may write the unit tangents to these two curves as

$$\frac{d\mathbf{r}}{ds} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{ds} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{ds} \quad \text{and} \quad \frac{\delta \mathbf{r}}{\delta s} = \frac{\partial \mathbf{r}}{\partial u}\frac{\delta u}{\delta s} + \frac{\partial \mathbf{r}}{\partial v}\frac{\delta v}{\delta s}$$

Now, if the vector dot-product of the unit vectors $\dfrac{d\mathbf{r}}{ds}, \dfrac{\delta \mathbf{r}}{\delta s}$ is zero then the two vectors are orthogonal. Using equations (109) the dot-product is

$$\begin{aligned}
\frac{d\mathbf{r}}{ds} \bullet \frac{\delta \mathbf{r}}{\delta s} &= \left(\frac{\partial \mathbf{r}}{\partial u}\frac{du}{ds} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{ds}\right) \bullet \left(\frac{\partial \mathbf{r}}{\partial u}\frac{\delta u}{\delta s} + \frac{\partial \mathbf{r}}{\partial v}\frac{\delta v}{\delta s}\right) \\
&= \frac{\partial \mathbf{r}}{\partial u}\frac{du}{ds} \bullet \frac{\partial \mathbf{r}}{\partial u}\frac{\delta u}{\delta s} + \frac{\partial \mathbf{r}}{\partial u}\frac{du}{ds} \bullet \frac{\partial \mathbf{r}}{\partial v}\frac{\delta v}{\delta s} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{ds} \bullet \frac{\partial \mathbf{r}}{\partial u}\frac{\delta u}{\delta s} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{ds} \bullet \frac{\partial \mathbf{r}}{\partial v}\frac{\delta v}{\delta s} \\
&= \left(\frac{\partial \mathbf{r}}{\partial u} \bullet \frac{\partial \mathbf{r}}{\partial u}\right)\frac{du}{ds}\frac{\delta u}{\delta s} + \left(\frac{\partial \mathbf{r}}{\partial u} \bullet \frac{\partial \mathbf{r}}{\partial v}\right)\left(\frac{du}{ds}\frac{\delta v}{\delta s} + \frac{dv}{ds}\frac{\delta u}{\delta s}\right) + \left(\frac{\partial \mathbf{r}}{\partial v} \bullet \frac{\partial \mathbf{r}}{\partial v}\right)\frac{dv}{ds}\frac{\delta v}{\delta s} \\
&= \left(\mathbf{r}_u \bullet \mathbf{r}_u\right)\frac{du}{ds}\frac{\delta u}{\delta s} + \left(\mathbf{r}_u \bullet \mathbf{r}_v\right)\left(\frac{du}{ds}\frac{\delta v}{\delta s} + \frac{dv}{ds}\frac{\delta u}{\delta s}\right) + \left(\mathbf{r}_v \bullet \mathbf{r}_v\right)\frac{dv}{ds}\frac{\delta v}{\delta s} \\
&= E\frac{du}{ds}\frac{\delta u}{\delta s} + F\left(\frac{du}{ds}\frac{\delta v}{\delta s} + \frac{dv}{ds}\frac{\delta u}{\delta s}\right) + G\frac{dv}{ds}\frac{\delta v}{\delta s} \\
&= \left\{E + F\left(\frac{du}{ds}\frac{\delta v}{\delta s} + \frac{dv}{ds}\frac{\delta u}{\delta s}\right)\frac{ds}{du}\frac{\delta s}{\delta u} + G\frac{dv}{ds}\frac{\delta v}{\delta s}\frac{ds}{du}\frac{\delta s}{\delta u}\right\}\frac{du}{ds}\frac{\delta u}{\delta s} \\
&= \left\{E + F\left(\frac{\delta v}{\delta u} + \frac{dv}{du}\right) + G\frac{dv}{du}\frac{\delta v}{\delta u}\right\}\frac{du}{ds}\frac{\delta u}{\delta s}
\end{aligned}$$

Using equations (145) and (146) we have

$$
\begin{aligned}
\frac{d\mathbf{r}}{ds}\bullet\frac{\delta\mathbf{r}}{\delta s} &= \left\{E + F\left(\lambda_1 + \lambda_2\right) + G\lambda_1\lambda_2\right\}\frac{du}{ds}\frac{\delta u}{\delta s} \\
&= \left\{E + F\left(\frac{GL - EN}{FN - GM}\right) + G\left(\frac{EM - FL}{FN - GM}\right)\right\}\frac{du}{ds}\frac{\delta u}{\delta s} \\
&= \left\{\frac{EFN - EGM + FGL - EFN + EGM - FGL}{FN - GM}\right\}\frac{du}{ds}\frac{\delta u}{\delta s} \\
&= 0
\end{aligned}
$$

Hence the unit vectors of the curves C_1 and C_2 are orthogonal as are the directions of principal curvatures λ_1 and λ_2.

If, at a point on a surface, the normal curvature κ_N is the same in every direction, then such points are known as umbilical points. In the directions of the parametric curves $u =$ constant $(du = 0)$ and $v =$ constant $(dv = 0)$, the normal curvatures are found from equation (137) as

$$\kappa_{du=0} = \frac{N}{G} \quad \text{and} \quad \kappa_{dv=0} = \frac{L}{E}$$

and in the direction of a curve where $du = dv$, equation (137) gives

$$\kappa_{du=dv} = \frac{L + 2M + N}{E + 2F + G}$$

Now, if the normal curvature is the same in every direction then we have two equations

$$
\begin{aligned}
\frac{L}{E} &= \frac{N}{G} \\
\frac{L}{E} &= \frac{L + 2M + N}{E + 2F + G}
\end{aligned}
\tag{149}
$$

From the second of equations (149) we have $EL + 2FL + GL = EL + 2EM + EN$ that simplifies to (a): $2FL - 2EM = EN - GL$. From the first of equations (149) $EN - GL = 0$, so substitution into (a) above gives $FL - EM = 0$ or $\frac{L}{E} = \frac{M}{F}$. Hence for an umbilical point, where the normal curvature is the same in every direction, the condition that holds

$$\frac{N}{G} = \frac{M}{F} = \frac{L}{E} \tag{150}$$

1.2.4 Surfaces of Revolution

In geodesy, the ellipsoid is a surface of revolution created by rotating an ellipse about its minor axis and is sometimes called an oblate ellipsoid. [A prolate ellipsoid is a surface of revolution created by rotating an ellipse about its major axis.] Some other surfaces of revolution are: a sphere (a circle rotated about a diameter), a cone (excluding the base), and a cylinder (excluding the ends).

The x,y,z Cartesian coordinates of a general surface of revolution having u,v curvilinear coordinates can be expressed in the general form

$$\begin{aligned} x(u,v) &= g(u)\cos v \\ y(u,v) &= g(u)\sin v \\ z(u,v) &= h(u) \end{aligned} \tag{151}$$

where $g(u)$, $h(u)$ are certain functions of u. A point on the surface of revolution has the position vector

$$\mathbf{r} = \mathbf{r}(u,v) = g\cos v\,\mathbf{i} + g\sin v\,\mathbf{j} + h\,\mathbf{k} \tag{152}$$

The derivatives of the position vector are

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial}{\partial u}\mathbf{r}(u,v) = g'\cos v\,\mathbf{i} + g'\sin v\,\mathbf{j} + h'\,\mathbf{k} \\ \mathbf{r}_v &= \frac{\partial}{\partial v}\mathbf{r}(u,v) = -g\sin v\,\mathbf{i} + g\cos v\,\mathbf{j} + 0\,\mathbf{k} \\ \mathbf{r}_{uv} &= \frac{\partial}{\partial v}\mathbf{r}_u = -g'\sin v\,\mathbf{i} + g'\cos v\,\mathbf{j} + 0\,\mathbf{k} \\ \mathbf{r}_{vu} &= \frac{\partial}{\partial u}\mathbf{r}_v = -g'\sin v\,\mathbf{i} + g'\cos v\,\mathbf{j} + 0\,\mathbf{k} \\ \mathbf{r}_{uu} &= \frac{\partial}{\partial u}\mathbf{r}_u = g''\cos v\,\mathbf{i} + g''\sin v\,\mathbf{j} + h''\,\mathbf{k} \\ \mathbf{r}_{vv} &= \frac{\partial}{\partial v}\mathbf{r}_v = -g\cos v\,\mathbf{i} - g\sin v\,\mathbf{j} + 0\,\mathbf{k} \end{aligned} \tag{153}$$

where $g' = \frac{d}{du}g(u)$; $g'' = \frac{d}{du}g'(u)$ and $h' = \frac{d}{du}h(u)$; $h'' = \frac{d}{du}h'(u)$

Using equations (109) and (153) gives the First Fundamental Coefficients as

$$\begin{aligned} E &= \mathbf{r}_u \bullet \mathbf{r}_u &&= g'^2 + h'^2 \\ F &= \mathbf{r}_u \bullet \mathbf{r}_v &&= 0 \\ G &= \mathbf{r}_v \bullet \mathbf{r}_v &&= g^2 \end{aligned} \tag{154}$$

In equations (154), $F = 0$ which indicates that the parametric curves on a surface of revolution (the u-curves and v-curves) are orthogonal.

Equation (116) gives

$$J \equiv \left|\mathbf{r}_u \times \mathbf{r}_v\right| = \sqrt{EG - F^2} = |g|\sqrt{g'^2 + h'^2} \tag{155}$$

noting that the normal vector $\mathbf{N}$ is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \overset{(+)}{\mathbf{i}} & \overset{(-)}{\mathbf{j}} & \overset{(+)}{\mathbf{k}} \\ g'\cos v & g'\sin v & h' \\ -g\sin v & g\cos v & 0 \end{vmatrix} = -gh'\cos v\,\mathbf{i} - gh'\sin v\,\mathbf{j} + gg'\,\mathbf{k} \tag{156}$$

and the unit normal vector $\hat{\mathbf{N}}$ is

$$\hat{\mathbf{N}} = \frac{\mathbf{N}}{|\mathbf{N}|} = \frac{\mathbf{N}}{\left|\mathbf{r}_u \times \mathbf{r}_v\right|} = \frac{\mathbf{N}}{J} = -\frac{gh'}{J}\cos v\,\mathbf{i} - \frac{gh'}{J}\sin v\,\mathbf{j} + \frac{gg'}{J}\mathbf{k} \tag{157}$$

Using equations (125) with (153) and (157) gives the Second Fundamental Coefficients as

$$\begin{aligned} L &= \hat{\mathbf{N}} \bullet \mathbf{r}_{uu} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{J} \bullet \mathbf{r}_{uu} &&= \frac{g}{J}\left(g'h'' - g''h'\right) = \frac{g'h'' - g''h'}{\sqrt{g'^2 + h'^2}} \\ M &= \hat{\mathbf{N}} \bullet \mathbf{r}_{uv} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{J} \bullet \mathbf{r}_{uv} &&= 0 \\ N &= \hat{\mathbf{N}} \bullet \mathbf{r}_{vv} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{J} \bullet \mathbf{r}_{vv} &&= \frac{g^2 h'}{J} = \frac{gh'}{\sqrt{g'^2 + h'^2}} \end{aligned} \tag{158}$$

So, for a general surface of revolution, we have

$$F = 0 \quad \text{and} \quad M = 0 \tag{159}$$

Substituting these results into equation (137) gives the equation for normal curvature on a general surface of revolution as

$$\kappa_N = \frac{L\,(du)^2 + N\,(dv)^2}{E\,(du)^2 + G\,(dv)^2} \tag{160}$$

The normal curvatures along the parametric curves u = constant ($du = 0$) and v = constant ($dv = 0$) are denoted κ_1 and κ_2 respectively and

$$\begin{aligned} \kappa_1 &= \frac{N}{G} = \frac{h'}{g\left(g'^2 + h'^2\right)^{\frac{1}{2}}} \\ \kappa_2 &= \frac{L}{E} = \frac{g'h'' - g''h'}{\left(g'^2 + h'^2\right)^{\frac{3}{2}}} \end{aligned} \tag{161}$$

Figure 18 shows two points P and Q on a surface S separated by a very small arc ds. The parametric curves $u,\ u+du$ and $v,\ v+dv$ form a very small rectangle on the surface and ds can be considered as the hypotenuse of a plane right-angled triangle and we may write

$$\tan\alpha = \frac{\sqrt{G}\,dv}{\sqrt{E}\,du} \tag{162}$$

where α is azimuth; a positive clockwise angle measured from the v-curve ($v = \text{constant}$).

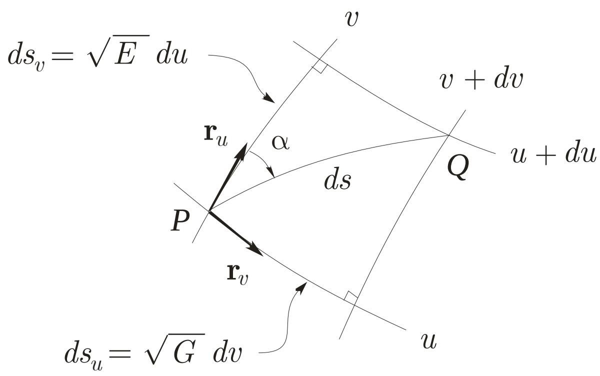


Figure 18: Small rectangle on surface S

Dividing the numerator and denominator of equation (160) by dv we may write the equation for normal curvature on a surface of revolution as

$$\kappa_N = \frac{L\left(\dfrac{du}{dv}\right)^2 + N}{E\left(\dfrac{du}{dv}\right)^2 + G} \tag{163}$$

and using equation (162) we have $\left(\dfrac{du}{dv}\right)^2 = \dfrac{G}{E\tan^2\alpha}$ and equation (163) can be written as

$$\kappa_N = \frac{\left(\dfrac{LG}{E\tan^2\alpha}\right) + N}{\left(\dfrac{EG}{E\tan^2\alpha}\right) + G} = \frac{\left(\dfrac{L}{E}\right) + \left(\dfrac{N}{G}\right)\tan^2\alpha}{1 + \tan^2\alpha}$$

Now since $1+\tan^2\alpha=\sec^2\alpha=\dfrac{1}{\cos^2\alpha}$ and $\tan^2\alpha=\dfrac{\sin^2\alpha}{\cos^2\alpha}$, we can write the normal curvature on a surface of revolution as

$$\kappa_N=\left(\frac{L}{E}\right)\cos^2\alpha+\left(\frac{N}{G}\right)\sin^2\alpha=\kappa_2\cos^2\alpha+\kappa_1\sin^2\alpha \tag{164}$$

Inspection of equation (164) reveals the following:

(i) the function κ_N must have optimum values, unless $\kappa_1=\kappa_2$. These optimum values are found by setting the derivative $\dfrac{d\kappa_N}{d\alpha}$ to zero. From equation (164)

$$\frac{d\kappa_N}{d\alpha}=2(\kappa_1-\kappa_2)\sin\alpha\cos\alpha=(\kappa_1-\kappa_2)\sin 2\alpha=0$$

and if $\kappa_1-\kappa_2\neq 0$ then the optimum values are where $\sin 2\alpha=0$, i.e., $2\alpha=n\pi$ or $\alpha=\frac{1}{2}n\pi$ where $n=0,1,2,3,\ldots$ $(\alpha=0^\circ,90^\circ,180^\circ,270^\circ,\ldots)$. This means that the optimum curvatures (of normal sections) are in the directions of the parametric curves on the surface of revolution.

(ii) there may be a point (or points) on the surface of revolution where $\kappa_1=\kappa_2$ in which case the curvature would be constant for any normal section in any direction. Such points are known as umbilic points. (For an ellipsoid representing the mathematical shape of the earth, the minor axis of the ellipsoid is the earth's polar axis and the north and south poles of the ellipsoid are umbilic points.)

With κ_α denoting the curvature of a normal section having the direction α on the surface, equation (164) becomes

$$\kappa_\alpha=\kappa_2\cos^2\alpha+\kappa_1\sin^2\alpha \tag{165}$$

And, since curvature (κ) is the reciprocal of radius of curvature (ρ) then

$$\frac{1}{\rho_\alpha}=\frac{1}{\rho_2}\cos^2\alpha+\frac{1}{\rho_1}\sin^2\alpha$$

and

$$\rho_\alpha=\frac{\rho_1\rho_2}{\rho_1\cos^2\alpha+\rho_2\sin^2\alpha} \tag{166}$$

This is *Euler's equation*[4] (on the curvature of surfaces) and gives the radius of curvature of a normal section in terms of the radii of curvature ρ_1, ρ_2 along the parametric curves $u =$ constant, $v =$ constant respectively and the azimuth α measured from the curve $v =$ constant. Note here that Euler's equation is applicable only on surfaces of revolution where the parametric curves are orthogonal and in the directions of principal curvature.

1.3 THE ELLIPSOID

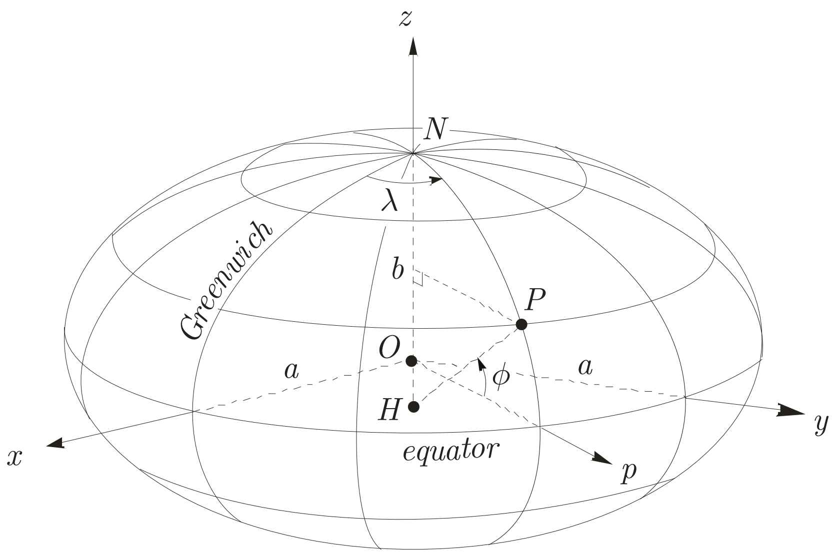


Figure 19: The reference ellipsoid

In geodesy, the ellipsoid is a surface of revolution created by rotating an ellipse (whose major and minor semi-axes lengths are a and b respectively and $a > b$) about its minor axis. The ϕ, λ curvilinear coordinate system is a set of orthogonal parametric curves on the surface – parallels of latitude ϕ and meridians of longitude λ with their respective

[4] This equation on the curvature of surfaces is named in honour of the great Swiss mathematician Leonard Euler (1707-1783) who, in a paper of 1760 titled *Recherches sur la courbure de surfaces* (Research on the curvature of surfaces), published the result $r = \dfrac{2fg}{f + g + (f - g)\cos 2\alpha}$ where f and g are extreme values of the radius of curvature r (Struik 1933). Using the trigonometric functions $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ and $\cos^2\alpha + \sin^2\alpha = 1$ his equation can be expressed in the form given above.

reference planes; the equator and the Greenwich meridian. Longitudes are measured 0° to ±180° (east positive, west negative) from the Greenwich meridian and latitudes are measured 0° to ±90° (north positive, south negative) from the equator. The *x,y,z* Cartesian coordinate system has an origin at *O*, the centre of the ellipsoid, and the z-axis is the minor axis (axis of revolution). The xOz plane is the Greenwich meridian plane (the origin of longitudes) and the xOy plane is the equatorial plane.

The positive x-axis passes through the intersection of the Greenwich meridian and the equator, the positive y-axis is advanced 90° east along the equator and the positive z-axis passes through the north pole of the ellipsoid. The Cartesian equation of the ellipsoid is

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{167}$$

where a and b are the semi-axes of the ellipsoid $(a > b)$.

All meridians of longitude on the ellipsoid are ellipses having semi-axes a and b $(a > b)$ since all meridian planes – e.g., Greenwich meridian plane xOz and the meridian plane pOz containing P – contain the z-axis of the ellipsoid and their curves of intersection are ellipses (planes intersecting surfaces create curves of intersection on the surface). This can be seen if we let $p^2 = x^2 + y^2$ in equation (167) which gives the familiar equation of the (meridian) ellipse

$$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1 \qquad (a < b) \tag{168}$$

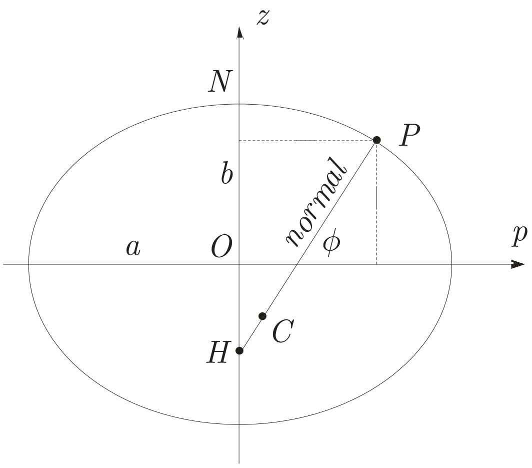


Figure 20: Meridian ellipse

In Figure 20, ϕ is the latitude of P (the angle between the equator and the normal), C is the centre of curvature and PC is the radius of curvature of the meridian ellipse at P. H is the intersection of the normal at P and the z-axis (axis of revolution).

All parallels of latitude on the ellipsoid are circles created by intersecting the ellipsoid with planes parallel to (or coincident with) the xOy equatorial plane. Replacing z with a constant C in equation (167) gives the equation for circular parallels of latitude

$$x^2 + y^2 = a^2\left(1 - \frac{C^2}{b^2}\right) = p^2 \qquad \left(0 \le C \le b;\;\; a > b\right) \tag{169}$$

All other curves on the surface of the ellipsoid created by intersecting the ellipsoid with a plane are ellipses. This can be demonstrated by using another set of coordinates x', y', z' that are obtained by a rotation of the x,y,z coordinates such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathbf{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{where} \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

where $\mathbf{R}$ is an orthogonal rotation matrix and $\mathbf{R}^{-1} = \mathbf{R}^T$ so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}^{-1}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

giving

$$\begin{aligned} x^2 &= r_{11}^2 x'^2 + r_{21}^2 y'^2 + r_{31}^2 z'^2 + 2r_{11}r_{21}x'y' + 2r_{11}r_{31}x'z' + 2r_{21}r_{31}y'z' \\ y^2 &= r_{12}^2 x'^2 + r_{22}^2 y'^2 + r_{32}^2 z'^2 + 2r_{12}r_{22}x'y' + 2r_{12}r_{32}x'z' + 2r_{22}r_{32}y'z' \\ z^2 &= r_{13}^2 x'^2 + r_{23}^2 y'^2 + r_{33}^2 z'^2 + 2r_{13}r_{23}x'y' + 2r_{13}r_{33}x'z' + 2r_{23}r_{33}y'z' \end{aligned}$$

$$\begin{aligned} x^2 + y^2 = {} & \left(r_{11}^2 + r_{12}^2\right)x'^2 + \left(r_{21}^2 + r_{22}^2\right)y'^2 + \left(r_{31}^2 + r_{32}^2\right)z'^2 + 2\left(r_{11}r_{21} + r_{12}r_{22}\right)x'y' \\ & + 2\left(r_{11}r_{31} + r_{12}r_{32}\right)x'z' + 2\left(r_{21}r_{31} + r_{22}r_{32}\right)y'z' \end{aligned}$$

Substituting into equation (167) gives the equation of the ellipsoid in x', y', z' coordinates

$$\begin{aligned} & \frac{1}{a^2}\left\{ \begin{aligned} & \left(r_{11}^2 + r_{12}^2\right)x'^2 + \left(r_{21}^2 + r_{22}^2\right)y'^2 + \left(r_{31}^2 + r_{32}^2\right)z'^2 + 2\left(r_{11}r_{21} + r_{12}r_{22}\right)x'y' \\ & + 2\left(r_{11}r_{31} + r_{12}r_{32}\right)x'z' + 2\left(r_{21}r_{31} + r_{22}r_{32}\right)y'z' \end{aligned} \right\} \\ & + \frac{1}{b^2}\left\{ r_{13}^2 x'^2 + r_{23}^2 y'^2 + r_{33}^2 z'^2 + 2r_{13}r_{23}x'y' + 2r_{13}r_{33}x'z' + 2r_{23}r_{33}y'z' \right\} = 1 \end{aligned} \tag{170}$$

In equation (170) let $z' = C_1$ where C_1 is a constant. The result will be the equation of a curve created by intersecting an inclined plane with the ellipsoid, i.e.,

$$\begin{aligned} &\left\{\frac{r_{11}^2 + r_{12}^2}{a^2} + \frac{r_{13}^2}{b^2}\right\} x'^2 + 2\left\{\frac{r_{11}r_{21} + r_{12}r_{22}}{a^2} + \frac{r_{13}r_{23}}{b^2}\right\} x'y' + \left\{\frac{r_{21}^2 + r_{22}^2}{a^2} + \frac{r_{23}^2}{b^2}\right\} y'^2 \\ &+ \left\{2C_1\left(r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33}\right)\right\} x' + \left\{2C_1\left(r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33}\right)\right\} y' \\ = &\ 1 - C_1^2\left\{r_{31}^2 + r_{32}^2 + r_{33}^2\right\} \end{aligned} \tag{171}$$

This equation can be expressed as

$$Ax'^2 + 2Hx'y' + By'^2 + Dx' + Ey' = 1 \tag{172}$$

where it can be shown that $AB - H^2 > 0$, hence it is the general Cartesian equation of an ellipse that is offset from the coordinate origin and rotated with respect to the coordinate axes (Grossman 1981). Equations of a similar form can be obtained for inclined planes $x' = C_2$ and $y' = C_3$, hence we may say, in general, inclined planes intersecting the ellipsoid will create curves of intersection that are ellipses.

Since the ellipsoid is a surface created by rotating an ellipse about its minor axis, then an ellipsoid can be completely defined by specifying one of the following pairs of parameters of the generating ellipse:

(i) (a,b) semi-major and semi-minor axes, or

(ii) (a,e^2) semi-major axis and eccentricity-squared, or

(iii) (a,f) semi-major axis and flattening.

a, b, e^2 and f are ellipsoid parameters (or constants) and they have been defined (for the ellipse) in earlier sections. Other parameters; c, e'^2 and n are also useful in developments to follow. They have also been defined in earlier sections as well as the interrelationship between all these parameters [see equations (22) to (37)]. In addition; the latitude functions W and V are useful in the computation of radius of curvature and Cartesian coordinates [see equations (43) to (47)] and the relationships between the normal at P and the axes of the ellipse are the same as for P on an ellipsoid [see Figure 9 and equations (48) to (51)]. Also latitudes ϕ, ψ and θ of P on an ellipse are identical to P on an ellipsoid and their interrelationship is often used to simplify formula [see Figure 9 and equations (54) to (56)]

1.3.1 Differential Geometry of the Ellipsoid

The x,y,z Cartesian coordinates are, using equations (43), (44) and the general notation of equations (151)

$$\begin{aligned} x(\phi,\lambda) &= \frac{a}{W}\cos\phi\cos\lambda = \frac{c}{V}\cos\phi\cos\lambda = g(\phi)\cos\lambda \\ y(\phi,\lambda) &= \frac{a}{W}\cos\phi\sin\lambda = \frac{c}{V}\cos\phi\sin\lambda = g(\phi)\sin\lambda \\ z(\phi,\lambda) &= \frac{a\left(1-e^2\right)}{W}\sin\phi = \frac{b}{V}\sin\phi = h(\phi) \end{aligned} \tag{173}$$

where

$$g(\phi) = g = \frac{c}{V}\cos\phi \quad \text{and} \quad h(\phi) = h = \frac{b}{V}\sin\phi \tag{174}$$

and, from equations (45) and equations (22) to (34)

$$\begin{aligned} &W^2 = 1 - e^2\sin^2\phi;\;\; V^2 = 1 + e'^2\cos^2\phi \;\;\text{and}\;\; c = \frac{a^2}{b} \;\;\text{with} \\ &f = \frac{a-b}{a};\;\; e^2 = \frac{a^2-b^2}{a^2} = f(2-f);\;\; e'^2 = \frac{a^2-b^2}{b^2} = \frac{f(2-f)}{(1-f)^2} \end{aligned} \tag{175}$$

Noting that $V = V(\phi)$ and $\dfrac{dV}{d\phi} = -\dfrac{e'^2}{V}\cos\phi\sin\phi$, the derivatives g', h', g'', h'' are

$$\begin{aligned} g' &= \frac{d}{d\phi}\left(\frac{c}{V}\cos\phi\right) = -\frac{c}{V^3}\sin\phi \\ h' &= \frac{d}{d\phi}\left(\frac{b}{V}\sin\phi\right) = \frac{c}{V^3}\cos\phi \\ g'' &= \frac{d}{d\phi}\left(-\frac{c}{V^3}\sin\phi\right) = \frac{c}{V^3}\cos\phi\left\{2 - 3\left(\frac{1+e'^2}{V^2}\right)\right\} \\ h'' &= \frac{d}{d\phi}\left(\frac{c}{V^3}\cos\phi\right) = \frac{c}{V^3}\sin\phi\left\{2 - \frac{3}{V^2}\right\} \end{aligned} \tag{176}$$

and related functions are

$$g'^2 + h'^2 = \left(\frac{c}{V^3}\right)^2 \quad \text{and} \quad g'h'' - g''h' = \left(\frac{c}{V^3}\right)^2 \tag{177}$$

First Fundamental Coefficients E, F and G are found from equations (154), (174) and (176) as

$$\begin{aligned} E &= g'^2 + h'^2 = \left(\frac{c}{V^3}\right)^2 \\ F &= 0 \\ G &= g^2 = \left(\frac{c}{V}\cos\phi\right)^2 \end{aligned} \tag{178}$$

and the related quantity J is found from equation (116)

$$J = \sqrt{EG - F^2} = \left(\frac{c}{V^2}\right)^2 \cos\phi \tag{179}$$

In equations (178), $F = 0$ which indicates (as we should expect) that the ϕ-curves (parallels of latitude) and λ-curves (meridians of longitude) are orthogonal.

Second Fundamental Coefficients L, M and N are found using equations (158), (174), (176) and (177) as

$$\begin{aligned} L &= \frac{g'h'' - g''h'}{\sqrt{g'^2 + h'^2}} = \frac{c}{V^3} \\ M &= 0 \\ N &= \frac{gh'}{\sqrt{g'^2 + h'^2}} = \frac{c}{V}\cos^2\phi \end{aligned} \tag{180}$$

Identical results for E,F,G and L,M,N can be obtained, with slightly less algebra, in the following manner.

The position vector of P on the surface of the ellipsoid is

$$\begin{aligned} \mathbf{r} = \mathbf{r}(\phi,\lambda) &= g\cos\lambda\,\mathbf{i} + g\sin\lambda\,\mathbf{j} + h\,\mathbf{k} \\ &= \frac{c}{V}\cos\phi\cos\lambda\,\mathbf{i} + \frac{c}{V}\cos\phi\sin\lambda\,\mathbf{j} + \frac{b}{V}\sin\phi\,\mathbf{k} \end{aligned}$$

and using equations (153), (174) and (176) the derivatives of the position vector are

$$\begin{aligned} \mathbf{r}_\phi &= g'\cos\lambda\,\mathbf{i} + g'\sin\lambda\,\mathbf{j} + h'\,\mathbf{k} \\ &= -\frac{c}{V^3}\sin\phi\cos\lambda\,\mathbf{i} - \frac{c}{V^3}\sin\phi\sin\lambda\,\mathbf{j} + \frac{c}{V^3}\cos\phi\,\mathbf{k} \\ \mathbf{r}_\lambda &= -g\sin\lambda\,\mathbf{i} + g\cos\lambda\,\mathbf{j} + 0\,\mathbf{k} \\ &= -\frac{c}{V}\cos\phi\sin\lambda\,\mathbf{i} + \frac{c}{V}\cos\phi\cos\lambda\,\mathbf{j} + 0\,\mathbf{k} \end{aligned}$$

The unit normal vector $\hat{\mathbf{N}}$ is found from equation (117) as

$$\hat{\mathbf{N}} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\lambda}{J}$$

and by equation (156) the cross product $\mathbf{r}_\phi \times \mathbf{r}_\lambda$ is

$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\lambda &= -gh' \cos v\,\mathbf{i} - gh' \sin v\,\mathbf{j} + gg'\,\mathbf{k}\\ &= -\left(\frac{c}{V^2}\right)^2 \cos^2\phi \cos\lambda\,\mathbf{i} - \left(\frac{c}{V^2}\right)^2 \cos^2\phi \sin\lambda\,\mathbf{j} - \left(\frac{c}{V^2}\right)^2 \cos\phi \sin\phi\,\mathbf{k}\end{aligned}$$

giving the unit normal vector to the surface of the ellipsoid as

$$\hat{\mathbf{N}} = -\cos\phi\cos\lambda\,\mathbf{i} - \cos\phi\sin\lambda\,\mathbf{j} - \sin\phi\,\mathbf{k}$$

The derivatives of the unit normal vector are

$$\begin{aligned}\hat{\mathbf{N}}_\phi &= \frac{\partial \hat{\mathbf{N}}}{\partial \phi} = \sin\phi\cos\lambda\,\mathbf{i} + \sin\phi\sin\lambda\,\mathbf{j} - \cos\phi\,\mathbf{k}\\ \hat{\mathbf{N}}_\lambda &= \frac{\partial \hat{\mathbf{N}}}{\partial \lambda} = \cos\phi\sin\lambda\,\mathbf{i} - \cos\phi\cos\lambda\,\mathbf{j} + 0\,\mathbf{k}\end{aligned}$$

Now using equations (109) the vectors above, the First Fundamental Coefficients *E, F* and *G* are

$$\begin{aligned}E &= \mathbf{r}_\phi \bullet \mathbf{r}_\phi\\ &= \left(\frac{c}{V^3}\right)^2 \sin^2\phi\cos^2\lambda + \left(\frac{c}{V^3}\right)^2 \sin^2\phi\sin^2\lambda + \left(\frac{c}{V^3}\right)^2 \cos^2\phi\\ &= \left(\frac{c}{V^3}\right)^2\\ F &= \mathbf{r}_\phi \bullet \mathbf{r}_\lambda\\ &= \frac{c^2}{V^4}\sin\phi\sin\lambda\cos\phi\cos\lambda - \frac{c^2}{V^4}\sin\phi\sin\lambda\cos\phi\cos\lambda\\ &= 0\\ G &= \mathbf{r}_\lambda \bullet \mathbf{r}_\lambda\\ &= \left(\frac{c}{V}\right)^2 \cos^2\phi\sin^2\lambda + \left(\frac{c}{V}\right)^2 \cos^2\phi\cos^2\lambda\\ &= \left(\frac{c}{V}\cos\phi\right)^2\end{aligned}$$

and using equations (124) the Second Fundamental Coefficients *L, M* and *N* are

$$
\begin{aligned}
L &= -\mathbf{r}_\phi \bullet \hat{\mathbf{N}}_\phi \\
&= \frac{c}{V^3}\sin^2\phi\cos^2\lambda + \frac{c}{V^3}\sin^2\phi\sin^2\lambda + \frac{c}{V^3}\cos^2\phi \\
&= \frac{c}{V^3} \\
2M &= -\left(\mathbf{r}_\phi \bullet \hat{\mathbf{N}}_\lambda + \mathbf{r}_\lambda \bullet \hat{\mathbf{N}}_\phi\right) \\
&= -\left(-\frac{c}{V^3}\sin\phi\cos\phi\sin\lambda\cos\lambda + \frac{c}{V^3}\sin\phi\cos\phi\sin\lambda\cos\lambda\right. \\
&\qquad \left. -\frac{c}{V}\sin\phi\cos\phi\sin\lambda\cos\lambda + \frac{c}{V}\sin\phi\cos\phi\sin\lambda\cos\lambda\right) \\
&= 0 \\
N &= -\mathbf{r}_\lambda \bullet \hat{\mathbf{N}}_\lambda \\
&= \frac{c}{V}\cos^2\phi\sin^2\lambda + \frac{c}{V}\cos^2\phi\cos^2\lambda \\
&= \frac{c}{V}\cos^2\phi
\end{aligned}
$$

These results are identical to those in equations (178) and (180)

The element of arc length ds on the surface of the ellipsoid is found from equations (108) and (178) as

$$
\begin{aligned}
(ds)^2 &= E\,(d\phi)^2 + 2F\,d\phi\,d\lambda + G\,(d\lambda)^2 \\
&= \left(\frac{c}{V^3}\right)^2 (d\phi)^2 + \left(\frac{c}{V}\cos\phi\right)^2 (d\lambda)^2
\end{aligned}
\tag{181}
$$

The element of arc length ds_λ along the λ-curve (a meridian of longitude where $\lambda =$ constant and $d\lambda = 0$) and the element of arc length ds_ϕ along the ϕ-curve (a parallel of latitude where $\phi =$ constant and $d\phi = 0$) are given by equations (121) as

$$
\begin{aligned}
ds_\lambda &= \sqrt{E}\,d\phi = \frac{c}{V^3}\,d\phi \\
ds_\phi &= \sqrt{G}\,d\lambda = \frac{c}{V}\cos\phi\,d\lambda
\end{aligned}
\tag{182}
$$

The angle θ between the tangents to the meridian of longitude and parallel of latitude curves are given implicitly by equations (120) as

$$
\cos\theta = \frac{F}{\sqrt{EG}} = 0 \quad \text{and} \quad \sin\theta = \frac{J}{\sqrt{EG}} = 1 \tag{183}
$$

and as we should expect, $\theta = 90^\circ$, since the meridians and parallels (λ-curves and ϕ-curves) form an orthogonal net (since, from above, $F = 0$).

The area of an infinitesimally small rectangle on the surface of the ellipsoid bounded by meridians λ and $\lambda + d\lambda$, and parallels ϕ and $\phi + d\phi$ is given by equation (122) as

$$dA = J\,d\phi\,d\lambda = \left(\frac{c}{V^2}\right)^2 \cos\phi\,d\phi\,d\lambda \tag{184}$$

The normal curvature on the ellipsoid is obtained from equation (160) as

$$\kappa_N = \frac{L(d\phi)^2 + N(d\lambda)^2}{E(d\phi)^2 + G(d\lambda)^2} = \frac{\frac{c}{V^3}(d\phi)^2 + \frac{c}{V}\cos^2\phi\,(d\lambda)^2}{\left(\frac{c}{V^3}\right)^2 (d\phi)^2 + \left(\frac{c}{V}\cos\phi\right)^2 (d\lambda)^2} \tag{185}$$

The optimum normal curvatures are $\kappa_1 = \frac{N}{G}$ and $\kappa_2 = \frac{L}{E}$, and these are in the directions of the parametric curves ϕ = constant $(d\phi = 0)$ and λ = constant $(d\lambda = 0)$ respectively.

From equations (161), (178) and (180) we have

$$\kappa_1 = \frac{N}{G} = \frac{\frac{c}{V}\cos^2\phi}{\left(\frac{c}{V}\right)^2 \cos^2\phi} = \frac{1}{\left(\frac{c}{V}\right)} \qquad \text{and} \qquad \kappa_2 = \frac{L}{E} = \frac{\frac{c}{V^3}}{\left(\frac{c}{V^3}\right)^2} = \frac{1}{\left(\frac{c}{V^3}\right)} \tag{186}$$

and since radius of curvature ρ is the reciprocal of the curvature κ, the optimum radii of curvature of the principal normal sections of the ellipsoid are

$$\rho_1 = \frac{c}{V} \qquad \text{and} \qquad \rho_2 = \frac{c}{V^3} \tag{187}$$

These optimum radii are the principal radii of curvature and are in the directions of the parametric curves; parallels of latitude (ϕ = constant) and meridians of longitude (λ = constant) respectively. These directions are the principal directions of the ellipsoid.

In general, at a point on an ellipsoid where $\phi \neq 90^\circ$ and $0 < \cos\phi \leq 1$ the quantity $V > 1$ and $V^3 > V$; and since c = constant for any particular ellipsoid then $\frac{c}{V} > \frac{c}{V^3}$. Hence the principal radii of curvature at a point are:

$\rho_1 = \frac{c}{V}$, that is the largest radius of curvature of a normal section and this normal section is in the direction of the parallel of latitude. In geodesy, this is known as the prime vertical normal section.

$\rho_2 = \dfrac{c}{V^3}$, that is the smallest radius of curvature of a normal section and this normal section is in the direction of the meridian of longitude. In geodesy, this is known as the meridian normal section.

These principal radii of curvature are designated ν and ρ respectively and so with equations (47)

$$\begin{aligned} \nu &= \frac{c}{V} = \frac{a}{W} && \text{(radius of curvature of prime vertical normal section)} \\ \rho &= \frac{c}{V^3} = \frac{a\left(1-e^2\right)}{W^3} && \text{(radius of curvature of meridian normal section)} \end{aligned} \tag{188}$$

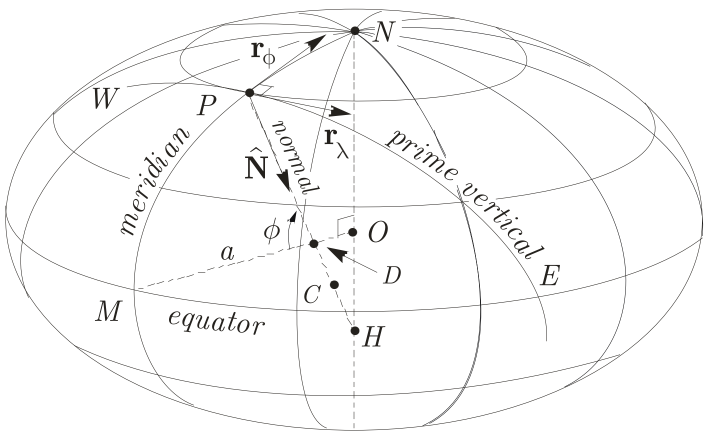


Figure 21: Meridian and prime vertical normal sections

In Figure 21 $\mathbf{r}_\phi$ and $\mathbf{r}_\lambda$ are orthogonal unit vectors in the directions of the parametric curves (meridians and parallels) and $\hat{\mathbf{N}}$ is the unit normal to the surface. The meridian normal section plane *MPNO* and the prime vertical normal section plane *WPEH* are orthogonal and contain the unit vectors $\mathbf{r}_\phi$ and $\mathbf{r}_\lambda$ respectively. Both normal section planes contain $\hat{\mathbf{N}}$ which lies along the normal to the surface that intersects the equatorial plane at *D* and the polar axis at *H*. *C* lies on the normal and is the centre of curvature of the meridian ellipse; $PC = \rho$ is the radius of curvature of the meridian section. *H* is the centre of curvature of the prime vertical ellipse and $PH = \nu$ is the radius of curvature of the prime vertical section. The distance $OH = \nu e^2 \sin\phi$. These ellipse relationships have been established previously, see equations (48), (51) and (66).

For <u>umbilical points</u> the following condition is obtained from equation (150)

$$\frac{G}{N}=\frac{F}{M}=\frac{E}{L} \quad \text{or} \quad \frac{\left(\frac{c}{V}\right)^2\cos^2\phi}{\frac{c}{V}\cos^2\phi}=\frac{0}{0}=\frac{\left(\frac{c}{V}\right)^2}{\frac{c}{V}}$$

which implies that $V^3 = V$ or $V^2 = 1$. Now; $V^2 = 1 + e'^2\cos^2\phi$ is unity only at the north and south poles of the ellipsoid where $\phi = \pm 90^\circ$. Hence both the <u>north and south poles of the ellipsoid are umbilical points</u>.

At the poles $(\phi = \pm 90^\circ)$

$$\rho_{pole} = \nu_{pole} = c = \frac{a^2}{b} \tag{189}$$

and <u>c is the polar radius of curvature</u>

At the equator $(\phi = 0^\circ)$

$$\rho_{equator} = \frac{c}{V^3} = \frac{c}{\left(1+e'^2\right)^{\frac{3}{2}}} = \frac{b^2}{a} \qquad \text{and} \qquad \nu_{equator} = \frac{c}{V} = \frac{c}{\left(1+e'^2\right)^{\frac{1}{2}}} = a \tag{190}$$

The <u>radius of curvature of a normal section having azimuth</u> α is ρ_α and from *Euler's equation* [equation (166)]

$$\rho_\alpha = \frac{\rho\nu}{\rho\sin^2\alpha + \nu\cos^2\alpha} \tag{191}$$

The <u>mean radius of curvature</u> ρ_m is the mean value of the radii of curvature for all values of $0 \le \alpha \le 2\pi$ in equation (191) where in general the mean value of a function $f(x)$ between $x = a$ and $x = b$ is

$$f_{mean} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

hence

$$\rho_m = \frac{1}{2\pi}\int_0^{2\pi}\rho_\alpha\,d\alpha = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\rho\nu}{\rho\sin^2\alpha+\nu\cos^2\alpha}d\alpha \qquad \text{since } 2\pi = 4\left(\frac{\pi}{2}\right)$$

dividing the numerator and denominator of the integrand by $\nu\cos^2\alpha$ and taking $\sqrt{\rho\nu}$ out as a constant gives

$$\rho_m = \frac{2\sqrt{\rho\nu}}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sec^2\alpha\sqrt{\dfrac{\rho}{\nu}}}{1+\dfrac{\rho}{\nu}\tan^2\alpha}\,d\alpha$$

With the substitution $u = \tan\alpha\sqrt{\dfrac{\rho}{\nu}}$ then $du = \sec^2\alpha\sqrt{\dfrac{\rho}{\nu}}\,d\alpha$ and $u^2 = \dfrac{\rho}{\nu}\tan^2\alpha$, and with the limits $\alpha = 0, \frac{\pi}{2}$ corresponding with $u = 0, \infty$ then

$$\rho_m = \frac{2\sqrt{\rho\nu}}{\pi}\int_0^{\infty} \frac{1}{1+u^2}\,du$$

Using the standard integral result $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$

$$\rho_m = \frac{2\sqrt{\rho\nu}}{\pi}\left[\arctan u\right]_0^{\infty} = \frac{2\sqrt{\rho\nu}}{\pi}\left(\frac{\pi}{2} - 0\right)$$

giving the mean radius of curvature as

$$\rho_m = \sqrt{\rho\nu} \tag{192}$$

The <u>Average and Gaussian curvatures</u> are, using equations (147), (148), (186) and (188)

$$\begin{aligned}\text{Average curvature } &= \tfrac{1}{2}\left(\kappa_1 + \kappa_2\right) = \tfrac{1}{2}\left(\frac{1}{\rho} + \frac{1}{\nu}\right)\\ \text{Gaussian curvature } &= \kappa_1\kappa_2 = \frac{1}{\rho\nu}\end{aligned} \tag{193}$$

The <u>radius of a parallel of latitude</u>, a circle on the surface of the ellipsoid, is found from *Meusnier's theorem* [equation (136)] as

$$r_{parallel} = \nu\cos\phi \tag{194}$$

Figure 22 illustrates the use of *Meusnier's theorem* to determine the radius of the parallel of latitude. At P on the ellipsoid, the parallel of latitude (radius r) and the prime vertical normal section (radius of curvature ν) have a <u>common</u> tangent vector $\mathbf{r}_\lambda$, and the plane containing the parallel of latitude and the prime vertical normal section plane make an angle of ϕ with each other. In Figure 22, $\hat{\mathbf{N}}$ is the unit normal to the surface and the distance $PH = \nu$ is the radius of curvature of the prime vertical section at P.

$\mathbf{r}_\phi$ is the unit tangent in the direction of the meridian normal section and $\mathbf{r}_\phi$ and $\mathbf{r}_\lambda$ are orthogonal unit vectors.

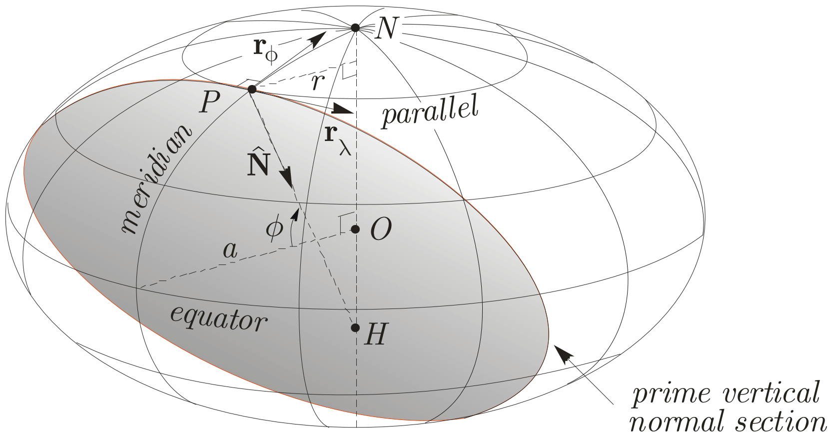


Figure 22: The prime vertical section and parallel of latitude

1.3.2 Meridian distance

Let m be the length of an arc of the meridian from the equator to a point in latitude ϕ then

$$dm = \rho\, d\phi \tag{195}$$

where ρ is the radius of curvature in the meridian plane and given by

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{\frac{3}{2}}} = \frac{a\left(1-e^2\right)}{W^3} \tag{196}$$

Alternatively, the radius of curvature in the meridian plane is also given by

$$\rho = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{3}{2}}} = \frac{c}{V^3} \tag{197}$$

Substituting equation (196) into equation (195) leads to series formula for the meridian distance m as a function of latitude ϕ and powers of e^2. Substituting equation (197) into equation (195) leads to series formula for m as a function of ϕ and powers of an ellipsoid constant $n = \dfrac{a-b}{a+b}$.

Series formula involving powers of e^2 are more commonly found in the geodetic literature but, series formula involving powers of n are more compact; and they are easier to "reverse", i.e., given m as a function of latitude ϕ and powers of n develop a series formula (by reversion of a series) that gives ϕ as a function m. This is very useful in the conversion of Universal Transverse Mercator (UTM) projection coordinates E,N to geodetic coordinates ϕ, λ.

Meridian distance as a series formula in powers of e^2

Using equations (195), (196) and (45) the meridian distance is given by the integral

$$m = \int_0^\phi \frac{a\left(1-e^2\right)}{W^3} d\phi = a\left(1-e^2\right)\int_0^\phi \frac{1}{W^3}\, d\phi = a\left(1-e^2\right)\int_0^\phi \left(1-e^2\sin^2\right)^{-\frac{3}{2}} d\phi \qquad (198)$$

This is an elliptic integral of the second kind that cannot be evaluated directly; instead, the integrand $\frac{1}{W^3} = \left(1-e^2\sin^2\phi\right)^{-\frac{3}{2}}$ is expanded in a series and then evaluated by term-by-term integration.

The integrand $\frac{1}{W^3} = \left(1-e^2\sin^2\phi\right)^{-\frac{3}{2}}$ can be expanded by use of the binomial series

$$(1+x)^\beta = \sum_{n=0}^{\infty} B_n^\beta\, x^n \qquad (199)$$

An infinite series where n is a positive integer, β is any real number and the binomial coefficients B_n^β are given by

$$B_n^\beta = \frac{\beta(\beta-1)(\beta-2)(\beta-3)\cdots(\beta-n+1)}{n!} \qquad (200)$$

The binomial series (199) is convergent when $-1 < x < 1$. In equation (200) $n!$ denotes n-factorial and $n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$. zero-factorial is defined as $0! = 1$ and the binomial coefficient $B_0^\beta = 1$.

In the case where β is a positive integer, say k, the binomial series (199) can be expressed as the finite sum

$$(1+x)^k = \sum_{n=0}^{k} B_n^k\, x^n \qquad (201)$$

where the binomial coefficients B_n^k in series (201) are given by

$$B_n^k = \frac{k!}{n!(k-n)!} \tag{202}$$

The binomial series is an important tool in geodesy where, as we shall see, it is used in the development of various solutions to geodetic problems. The invention of the binomial series is attributed to Isaac Newton, who, in a letter to the German mathematician Gottfried Leibniz in June 1676, set out his theorem as:

"The Extraction of Roots are much shortened by the Theorem

$$(P+PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c \tag{a}$$

where $P+PQ$ stands for a Quantity whose Root or Power or whose Root of a Power is to be found, P being the first Term of that quantity, Q being the remaining terms divided by the first term, and m/n the numerical Index of the power of $P+PQ$. This may be a Whole Number or (so to speak) a Broken Number; a positive number or a negative one." [Newton then uses several cases to describe how P, Q, m and n are obtained and then defines A, B, C and D] "In this last case, if $(a^3+b^2x)^{-2/3}$ to be taken to mean $(P+PQ)^{-2/3}$ in the Formula, then $P=a^3$, $Q=b^2x/a^3$, $m=-2$, $n=3$. Finally, in place of the terms that occur in the course of the work in the Quotient, I shall use A, B, C, D, &c. Thus A stands for the first term $P^{m/n}$; B for the second term $\frac{m}{n}AQ$; and so on. The use of this Formula will become clear through Examples." [The examples show the application of the formula in cases in which the exponents are $1/2$, $1/5$, $-1/3$, $4/3$, 5, -1, $-3/5$]

In a subsequent letter to Leibniz in October 1676, Newton explains in some detail how he made his early discoveries, and discloses that his binomial rule was formulated twelve years earlier, in 1664, while he was an undergraduate at Cambridge University (Newman 1956).

Letting $P=a$, $Q=\frac{x}{a}$ and $\beta=\frac{m}{n}$ in Newton's formula (a) above gives

$$(a+x)^\beta = a^\beta + \beta a^{\beta-1}x + \frac{\beta(\beta-1)}{2!}a^{\beta-2}x^2 + \frac{\beta(\beta-1)(\beta-2)}{3!}a^{\beta-3}x^3 + \cdots \tag{b}$$

Letting $a=1$ in equation (b) will give the expanded form of equation (199).

Letting $a=1$ and setting $\beta=k$ a positive integer in equation (b) will give the expanded form of equation (201)

The series (199) for $\beta = -\frac{3}{2}$ is

$$(1+x)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} B_n^{-\frac{3}{2}} x^n = B_0^{-\frac{3}{2}} x^0 + B_1^{-\frac{3}{2}} x^1 + B_2^{-\frac{3}{2}} x^2 + B_3^{-\frac{3}{2}} x^3 + \cdots \tag{203}$$

The binomial coefficients $B_n^{-\frac{3}{2}}$ for the series (199) are given by equation (200) as

$$n = 0 \qquad B_0^{-\frac{3}{2}} = 1$$

$$n = 1 \qquad B_1^{-\frac{3}{2}} = \frac{\left(-\frac{3}{2}\right)}{1!} = -\frac{3}{2}$$

$$n = 2 \qquad B_2^{-\frac{3}{2}} = \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2!} = \frac{3 \cdot 5}{2 \cdot 4}$$

$$n = 3 \qquad B_3^{-\frac{3}{2}} = \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{3!} = -\frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6}$$

Inspecting the results above, we can see that the binomial coefficients $B_n^{-\frac{3}{2}}$ form a sequence

$$1,\ -\frac{3}{2},\ \frac{3 \cdot 5}{2 \cdot 4},\ -\frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6},\ \frac{3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8},\ -\frac{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10},\ \cdots$$

Using these coefficients gives (Baeschlin 1948, p.48; Jordan/Eggert/Kneissl 1958, p.75; Rapp 1982, p.26)

$$\begin{aligned} \frac{1}{W^3} = \left(1 - e^2 \sin^2 \phi\right)^{-\frac{3}{2}} = 1 &+ \frac{3}{2} e^2 \sin^2 \phi + \frac{3 \cdot 5}{2 \cdot 4} e^4 \sin^4 \phi + \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6} e^6 \sin^6 \phi \\ &+ \frac{3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8} e^8 \sin^8 \phi + \frac{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} e^{10} \sin^{10} \phi + \cdots \end{aligned} \tag{204}$$

To simplify this expression, and make the eventual integration easier, the powers of $\sin \phi$ can be expressed in terms of multiple angles using the standard form

$$\begin{aligned} \sin^{2n} \phi = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \Bigg\{ &\cos 2n\phi - \binom{2n}{1} \cos(2n-2)\phi + \binom{2n}{2} \cos(2n-4)\phi \\ &- \binom{2n}{3} \cos(2n-6)\phi + \cdots (-1)^n \binom{2n}{n-1} \cos 2\phi \Bigg\} \end{aligned} \tag{205}$$

Using equation (205) and the binomial coefficients $B_n^{2n} = \binom{2n}{n}$ computed using equation (202) gives

$$\sin^2 \phi = \frac{1}{2} - \frac{1}{2} \cos 2\phi$$

$$\sin^4\phi = \frac{3}{8} + \frac{1}{8}\cos 4\phi - \frac{1}{2}\cos 2\phi$$

$$\sin^6\phi = \frac{5}{16} - \frac{1}{32}\cos 6\phi + \frac{3}{16}\cos 4\phi - \frac{15}{32}\cos 2\phi$$

$$\sin^8\phi = \frac{35}{128} + \frac{1}{128}\cos 8\phi - \frac{1}{16}\cos 6\phi + \frac{7}{32}\cos 4\phi - \frac{7}{16}\cos 2\phi$$

$$\sin^{10}\phi = \frac{63}{256} - \frac{1}{512}\cos 10\phi + \frac{5}{256}\cos 8\phi - \frac{45}{512}\cos 6\phi + \frac{15}{64}\cos 4\phi - \frac{105}{256}\cos 2\phi \qquad (206)$$

Substituting equations (206) into equation (204) and arranging according to $\cos 2\phi$, $\cos 4\phi$, etc., we obtain (Baeschlin 1948, p.48; Jordan/Eggert/Kneissl 1958, p.75; Rapp 1982, p.27)

$$\frac{1}{W^3} = \left(1 - e^2\sin^2\phi\right)^{-\frac{3}{2}} = A - B\cos 2\phi + C\cos 4\phi - D\cos 6\phi + E\cos 8\phi - F\cos 10\phi + \cdots \qquad (207)$$

where the coefficients A, B, C, etc., are

$$\begin{aligned}
A &= 1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \frac{175}{256}e^6 + \frac{11025}{16384}e^8 + \frac{43659}{65536}e^{10} + \cdots \\
B &= \frac{3}{4}e^2 + \frac{15}{16}e^4 + \frac{525}{512}e^6 + \frac{2205}{2048}e^8 + \frac{72765}{65536}e^{10} + \cdots \\
C &= \frac{15}{64}e^4 + \frac{105}{256}e^6 + \frac{2205}{4096}e^8 + \frac{10395}{16384}e^{10} + \cdots \\
D &= \frac{35}{512}e^6 + \frac{315}{2048}e^8 + \frac{31185}{131072}e^{10} + \cdots \\
E &= \frac{315}{16384}e^8 + \frac{3465}{65536}e^{10} + \cdots \\
F &= \frac{693}{131072}e^{10} + \cdots
\end{aligned} \qquad (208)$$

Substituting equation (207) into equation (198) gives the meridian distance as

$$m = a\left(1 - e^2\right)\int_0^{\phi}\left\{A - B\cos 2\phi + C\cos 4\phi - D\cos 6\phi + E\cos 8\phi - F\cos 10\phi + \cdots\right\}d\phi$$

Integrating term-by-term using the standard integral result $\int_0^x \cos ax\,dx = \frac{\sin ax}{a}$ gives the meridian distance m from the equator to a point in latitude ϕ as

$$m = a\left(1-e^2\right)\left\{A\phi - \frac{B}{2}\sin 2\phi + \frac{C}{4}\sin 4\phi - \frac{D}{6}\sin 6\phi + \frac{E}{8}\sin 8\phi - \frac{F}{10}\sin 10\phi + \cdots\right\} \tag{209}$$

where ϕ is in radians and the coefficients $A, B, C,$ etc., are given by equations (208).

From equation (209), the quadrant distance Q[5], the meridian distance from the equator to the pole, is

$$Q = a\left(1-e^2\right)A\left(\tfrac{1}{2}\pi\right) \tag{210}$$

Equation (209) may be simplified by multiplying the coefficients by $\left(1-e^2\right)$ and expressing the meridian distance as

$$m = a\left\{A_0\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi - A_{10}\sin 10\phi + \cdots\right\} \tag{211}$$

where $A_0 = \left(1-e^2\right)A$, $A_2 = \left(1-e^2\right)\dfrac{B}{2}$, $A_4 = \left(1-e^2\right)\dfrac{C}{4}$, etc., and

$$\begin{aligned}
A_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16384}e^8 - \frac{441}{65536}e^{10} + \cdots \\
A_2 &= \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 + \frac{35}{512}e^8 + \frac{735}{16384}e^{10} + \cdots\right) \\
A_4 &= \frac{15}{256}\left(e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 + \frac{105}{256}e^{10} + \cdots\right) \\
A_6 &= \frac{35}{3072}\left(e^6 + \frac{5}{4}e^8 + \frac{315}{256}e^{10} + \cdots\right) \\
A_8 &= \frac{315}{131072}\left(e^8 + \frac{7}{4}e^{10} + \cdots\right) \\
A_{10} &= \frac{693}{131072}\left(e^{10} + \cdots\right)
\end{aligned} \tag{212}$$

[5] The quadrant distance is the length of the meridian arc from the equator to the pole and the ten-millionth part of this distance was originally intended to have defined the metre when that unit was introduced. For those interested in the history of geodesy, *The Measure Of All Things* (Adler 2002) has a detailed account of the measurement of the French Arc (an arc of the meridian from Dunkerque, France to Barcelona, Spain and passing through Paris) by John-Baptiste-Joseph Delambre and Pierre-François-André Méchain in 1792-9 during the French Revolution. The analysis of their measurements enabled the computation of the dimensions of the earth that lead to the definitive metre platinum bar of 1799.

The GDA Technical Manual formula for meridian distance

In the *Geocentric Datum of Australia Technical Manual* (ICSM 2002) the formula for meridian distance is given in the form

$$m = a\left\{B_0\phi - B_2\sin 2\phi + B_4\sin 4\phi - B_6\sin 6\phi\right\} \tag{213}$$

where

$$\begin{aligned} B_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 \\ B_2 &= \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6\right) \\ B_4 &= \frac{15}{256}\left(e^4 + \frac{3}{4}e^6\right) \\ B_6 &= \frac{35}{3072}e^6 \end{aligned} \tag{214}$$

This is a contraction of equation (211) and the coefficients B_0, B_2, B_4 and B_6 exclude all terms involving powers of the eccentricity greater than e^6 in the coefficients A_0, A_2, A_4 and A_6. Equations (213) and (214) are the same formula given in Lauf (1983, p. 36, eq'n 3.55).

Meridian distance as a series expansion in powers of n

The German geodesist F.R. Helmert (1880) gave a series formula for meridian distance m as a function of latitude ϕ and powers of an ellipsoid constant n that requires fewer terms than the meridian distance formula involving powers of e^2.

Using equations (195) and (197) the differentially small meridian distance dm is given by

$$dm = \frac{c}{V^3}\,d\phi \tag{215}$$

With the ellipsoid constant n defined as

$$n = \frac{a-b}{a+b} = \frac{f}{2-f} \tag{216}$$

the following relationships can be derived

$$c = \frac{a^2}{b} = a\left(\frac{1+n}{1-n}\right), \quad e^2 = \frac{4n}{(1+n)^2}, \quad e'^2 = \frac{4n}{(1-n)^2} \tag{217}$$

Using the last of equations (217) we may write

$$V^2 = 1 + e'^2 \cos^2\phi = \frac{(1-n)^2 + 4n\cos^2\phi}{(1-n)^2}$$

and using the trigonometric relationship $\cos 2\phi = 2\cos^2\phi - 1$

$$\begin{aligned} V^2 &= \frac{(1-n)^2 + 2n\cos 2\phi + 2n}{(1-n)^2} \\ &= \frac{1}{(1-n)^2}\left(1 + n^2 + 2n\cos 2\phi\right) \end{aligned} \tag{218}$$

Now we can make use of *Euler's identities:* $e^{i\phi} = \cos\phi + i\sin\phi$, $e^{-i\phi} = \cos\phi - i\sin\phi$ in simplifying equation (218) Note that i is the imaginary unit $\left(i^2 = -1\right)$ and $e = 2.718281828\ldots$ is the base of the natural logarithms. e in Euler's identities should not be confused with the eccentricity of the ellipsoid.

Adding Euler's identities gives $2\cos\phi = e^{i\phi} + e^{-i\phi}$ and replacing ϕ with 2ϕ gives $2\cos 2\phi = e^{i2\phi} + e^{-i2\phi}$. Substituting this result into equation (218) gives

$$\begin{aligned} V^2 &= \frac{1}{(1-n)^2}\left(1 + n^2 + n\left(e^{i2\phi} + e^{-i2\phi}\right)\right) \\ &= \frac{1}{(1-n)^2}\left(1 + n^2 + ne^{i2\phi} + ne^{-i2\phi}\right) \\ &= \frac{1}{(1-n)^2}\left(1 + ne^{i2\phi}\right)\left(1 + ne^{-i2\phi}\right) \end{aligned}$$

Now an expression for $\frac{1}{V^3}$ in equation (215) can be developed as

$$\begin{aligned} \frac{1}{V^3} &= \left(V^2\right)^{-\frac{3}{2}} \\ &= \left((1-n)^{-2}\right)^{-\frac{3}{2}}\left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}}\left(1 + ne^{-i2\phi}\right)^{-\frac{3}{2}} \\ &= (1-n)^3\left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}}\left(1 + ne^{-i2\phi}\right)^{-\frac{3}{2}} \end{aligned} \tag{219}$$

Using equation (219) and the first of equations (217) in equation (215) gives

$$dm = a\left(\frac{1+n}{1-n}\right)(1-n)^3\left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}}\left(1 + ne^{-i2\phi}\right)^{-\frac{3}{2}} d\phi \tag{220}$$

Now $\left(\frac{1+n}{1-n}\right)(1-n)^3 = (1+n)(1-n)^2 = (1-n)\left(1-n^2\right)$ and equation (220) becomes (Lauf 1983, p. 36, eq'n 3.57)

$$dm = a(1-n)\left(1-n^2\right)\left(1+ne^{i2\phi}\right)^{-\frac{3}{2}}\left(1+ne^{-i2\phi}\right)^{-\frac{3}{2}} d\phi \tag{221}$$

Using the binomial series as previously developed [see equation (204)] we may write

$$\begin{aligned}\left(1+ne^{i2\phi}\right)^{-\frac{3}{2}} = 1 - \frac{3}{2}ne^{i2\phi} + \frac{3\cdot 5}{2\cdot 4}n^2 e^{i4\phi} - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}n^3 e^{i6\phi} \\ + \frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}n^4 e^{i8\phi} - \frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}n^5 e^{i10\phi} + \cdots\end{aligned}$$

and

$$\begin{aligned}\left(1+ne^{i2\phi}\right)^{-\frac{3}{2}} = 1 - \frac{3}{2}ne^{-i2\phi} + \frac{3\cdot 5}{2\cdot 4}n^2 e^{-i4\phi} - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}n^3 e^{-i6\phi} \\ + \frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}n^4 e^{-i8\phi} - \frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}n^5 e^{-i10\phi} + \cdots\end{aligned}$$

The product of these two series, after gathering terms, will be a series in terms $\left(e^{i2\phi} + e^{-i2\phi}\right) = 2\cos 2\phi$, $\left(e^{i4\phi} + e^{-i4\phi}\right) = 2\cos 4\phi$, $\left(e^{i6\phi} + e^{-i6\phi}\right) = 2\cos 6\phi$, etc.; each term having coefficients involving powers of n. Using this product in equation (221) and simplifying gives

$$\begin{aligned}dm = a(1-n)\left(1-n^2\right)\Big\{ & 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots \\ & -2\cos 2\phi \left(\frac{3}{2}n + \frac{45}{16}n^3 + \frac{525}{128}n^5 + \cdots\right) \\ & +2\cos 4\phi \left(\frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots\right) \\ & -2\cos 6\phi \left(\frac{35}{16}n^3 + \frac{945}{256}n^5 + \cdots\right) \\ & +2\cos 8\phi \left(\frac{315}{128}n^4 + \cdots\right) \\ & -2\cos 10\phi \left(\frac{693}{256}n^5 + \cdots\right) + \cdots \Big\} d\phi\end{aligned}$$

Integrating term-by-term using the standard integral result $\int_0^x \cos ax\,dx = \frac{\sin ax}{a}$ gives the meridian distance m from the equator to a point in latitude ϕ as

$$m = a(1-n)\left(1-n^2\right)\left\{a_0\phi - a_2\sin 2\phi + a_4\sin 4\phi - a_6\sin 6\phi + a_8\sin 8\phi - a_{10}\sin 10\phi + \cdots\right\} \tag{222}$$

where

$$\begin{aligned}
a_0 &= 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots \\
a_2 &= \frac{3}{2}n + \frac{45}{16}n^3 + \frac{525}{128}n^5 + \cdots \\
a_4 &= \frac{1}{2}\left(\frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots\right) \\
a_6 &= \frac{1}{3}\left(\frac{35}{16}n^3 + \frac{945}{256}n^5 + \cdots\right) \\
a_8 &= \frac{1}{4}\left(\frac{315}{128}n^4 + \cdots\right) \\
a_{10} &= \frac{1}{5}\left(\frac{693}{256}n^5 + \cdots\right)
\end{aligned} \tag{223}$$

Helmert's formula for meridian distance

Jordan/Eggert/Kneissl (1958, p.83) in a section titled *Helmertsche Formeln zur Rektifikation des Meridianbogens* (Helmert's formula for meridian distance) outlines a method of derivation attributed to Helmert (1880) that is similar to the derivation in the previous section. Their starting point (and presumably Helmert's) was $\rho = \frac{a\left(1-e^2\right)}{W^3}$ and $\left(1-e^2\right) = \frac{(1-n)^2}{(1+n)^2}$ rather than $\rho = \frac{c}{V^3}$ and $dm = \frac{c}{V^3}d\phi$ as above but the end result (Jordan/Eggert/Kneissl 1958, eq'n 38, p.83) is similar in form to equation (222) but without the term $-a_{10}\sin 10\phi$ and the coefficients exclude all terms involving powers of n greater than n^4. With these restrictions we give *Helmert's formula* as (Lauf 1983, p. 36, eq'n 3.55)

$$m = a(1-n)\left(1-n^2\right)\left\{b_0\phi - b_2\sin 2\phi + b_4\sin 4\phi - b_6\sin 6\phi + b_8\sin 8\phi - \cdots\right\} \tag{224}$$

where

$$\begin{aligned} b_0 &= 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots \\ b_2 &= \frac{3}{2}n + \frac{45}{16}n^3 + \cdots \\ b_4 &= \frac{1}{2}\left(\frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots\right) \\ b_6 &= \frac{1}{3}\left(\frac{35}{16}n^3 + \cdots\right) \\ b_8 &= \frac{1}{4}\left(\frac{315}{128}n^4 + \cdots\right) \end{aligned} \tag{225}$$

<u>An alternative form of *Helmert's formula*</u>

An alternative form of *Helmert's formula* [equation (224)] can be developed by noting that

$$\begin{aligned} (1-n)\left(1-n^2\right) &= \frac{1+n}{1+n}(1-n)\left(1-n^2\right) \\ &= \frac{\left(1-n^2\right)\left(1-n^2\right)}{1+n} \end{aligned}$$

Multiplying the coefficients $b_0,\ b_2,\ b_4,\ b_6$ and b_8 by $\left(1-n^2\right)\left(1-n^2\right)$ gives

$$m = \frac{a}{1+n}\left\{c_0\phi - c_2\sin 2\phi + c_4\sin 4\phi - c_6\sin 6\phi + c_8\sin 8\phi - \cdots\right\} \tag{226}$$

where

$$\begin{aligned} c_0 &= 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \cdots \\ c_2 &= \frac{3}{2}\left(n - \frac{1}{8}n^3 - \cdots\right) \\ c_4 &= \frac{15}{16}\left(n^2 - \frac{1}{4}n^4 - \cdots\right) \\ c_6 &= \frac{35}{48}\left(n^3 - \cdots\right) \\ c_8 &= \frac{315}{512}\left(n^4 - \cdots\right) \end{aligned} \tag{227}$$

Equation (226) with expressions for the coefficients c_0, c_2, c_4 etc., is, except for a slight change in notation, the same as Rapp (1982, p. 30, eq'n 95) who cites Helmert (1880) and is essentially the same as Baeschlin (1948, p. 50, eq'n 5.5) and Jordan/Eggert/Kneissl (1958, p.83-2, eq'ns 38 and 42)

Latitude from *Helmert's formula* by reversion of a series

Helmert's formula [equation (224)] gives meridian distance m as a function of latitude ϕ and powers of n and this formula (or another involving ϕ and e^2 developed above) is necessary for the conversion of ϕ, λ to UTM projection coordinates *E,N.* The reverse operation, *E,N* to ϕ, λ requires a method of computing ϕ given m. This could be done by a computer program implementing the Newton-Raphson scheme of iteration (described in a following section), or as it was in pre-computer days, by inverse interpolation of printed tables of latitudes and meridian distances. An efficient direct formula can be obtained by "reversing" *Helmert's formula* using *Lagrange's theorem* to give a series formula for ϕ as a function of an angular quantity σ and powers of *n;* and σ, as we shall see, is directly connected to the meridian distance m. We thus have a direct way of computing ϕ given m that is extremely useful in map projection computations.

The following pages contain an expanded explanation of the very concise derivation set out in Lauf (1983); the only text on Geodesy where (to our knowledge) this useful technique and formula is set down.

Using *Helmert's formula* [equation (224)] and substituting the value $\phi = \frac{1}{2}\pi$ gives a formula for the quadrant distance Q as

$$Q = a(1-n)\left(1-n^2\right)\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right)\frac{\pi}{2} \tag{228}$$

Also, we can establish two quantities:

(i) *G,* the mean length of a meridian arc of one radian

$$G = \frac{Q}{\frac{1}{2}\pi} = a(1-n)\left(1-n^2\right)\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right) \tag{229}$$

(ii) σ, an angular quantity in radians and

$$\sigma = \frac{m}{G} \tag{230}$$

An expression for σ as a function of ϕ and powers of n is obtained by dividing equation (229) into *Helmert's formula* [equation (224)] giving

$$\begin{aligned}\sigma &= \phi - \left\{\frac{\frac{3}{2}n + \frac{45}{16}n^3 + \cdots}{1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots}\right\}\sin 2\phi + \frac{1}{2}\left\{\frac{\frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots}{1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots}\right\}\sin 4\phi \\ &\quad - \frac{1}{3}\left\{\frac{\frac{35}{16}n^3 + \cdots}{1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots}\right\}\sin 6\phi + \frac{1}{4}\left\{\frac{\frac{315}{128}n^4 + \cdots}{1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots}\right\}\sin 8\phi - \cdots\end{aligned} \tag{231}$$

Using a special case of the binomial series [equation (199) with $\beta = -1$]

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

the numerator of each coefficient in the equation for σ can be written as

$$\begin{aligned}\left(1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots\right)^{-1} &= 1 - \left(\frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots\right) + \left(\frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots\right)^2 \\ &\quad - \left(\frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots\right)^3 + \cdots\end{aligned}$$

and expanding the right-hand side and simplifying gives

$$\left(1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots\right)^{-1} = 1 - \frac{9}{4}n^2 + \frac{99}{64}n^4 - \cdots \tag{232}$$

Substituting equation (232) into equation (231), multiplying the terms and simplifying gives the equation for σ as (Lauf 1983, p. 37, eq'n 3.67)

$$\begin{aligned}\sigma = \frac{m}{G} &= \phi - \left(\frac{3}{2}n - \frac{9}{16}n^3 - \cdots\right)\sin 2\phi + \left(\frac{15}{16}n^2 - \frac{15}{32}n^4 - \cdots\right)\sin 4\phi \\ &\quad - \left(\frac{35}{48}n^3 - \cdots\right)\sin 6\phi + \left(\frac{315}{512}n^4 - \cdots\right)\sin 8\phi - \cdots\end{aligned} \tag{233}$$

If we require the value of ϕ corresponding to a particular value of σ, then the series (233) needs to be reversed. This can be done using *Lagrange's theorem* (or *Lagrange's expansion*) a proof of which can be found in Carr (1970).

Suppose that

$$y = z + xF(y) \quad \text{or} \quad z = y - xF(y) \tag{234}$$

then *Lagrange's theorem* states that

$$\begin{aligned} f(y) = f(z) + xF(z)f'(z) + \frac{x^2}{2!}\frac{d}{dz}\left[\{F(z)\}^2 f'(z)\right] + \frac{x^3}{3!}\frac{d^2}{dz^2}\left[\{F(z)\}^3 f'(z)\right] + \cdots \\ \cdots + \frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\{F(z)\}^n f'(z)\right] \end{aligned} \tag{235}$$

In our case, comparing the variables in equations (233) and (234), $z = \sigma$, $y = \phi$ and $x = 1$, and if we choose $f(y) = y$ then $f(z) = z$ and $f'(z) = 1$. So, in our case equation (233) can be expressed as

$$\sigma = \phi - F(\phi) \tag{236}$$

and *Lagrange's theorem* gives

$$\phi = \sigma + F(\sigma) + \frac{1}{2}\frac{d}{d\sigma}\left[\{F(\sigma)\}^2\right] + \frac{1}{6}\frac{d^2}{d\sigma^2}\left[\{F(\sigma)\}^3\right] + \cdots + \frac{1}{n!}\frac{d^{n-1}}{d\sigma^{n-1}}\left[\{F(\sigma)\}^n\right] \tag{237}$$

Now, comparing equations (236) and (233) the function $F(\phi)$ is

$$\begin{aligned} F(\phi) = \left(\frac{3}{2}n - \frac{9}{16}n^3 - \cdots\right)\sin 2\phi - \left(\frac{15}{16}n^2 - \frac{15}{32}n^4 - \cdots\right)\sin 4\phi \\ + \left(\frac{35}{48}n^3 - \cdots\right)\sin 6\phi - \left(\frac{315}{512}n^4 - \cdots\right)\sin 8\phi - \cdots \end{aligned}$$

and so replacing ϕ with σ gives the function $F(\sigma)$ in equation (237) as

$$\begin{aligned} F(\sigma) = \left(\frac{3}{2}n - \frac{9}{16}n^3 - \cdots\right)\sin 2\sigma - \left(\frac{15}{16}n^2 - \frac{15}{32}n^4 - \cdots\right)\sin 4\sigma \\ + \left(\frac{35}{48}n^3 - \cdots\right)\sin 6\sigma - \left(\frac{315}{512}n^4 - \cdots\right)\sin 8\sigma - \cdots \end{aligned} \tag{238}$$

Squaring $F(\sigma)$ gives

$$\{F(\sigma)\}^2 = \left(\frac{9}{4}n^2 - \frac{27}{16}n^4 + \cdots\right)\sin^2 2\sigma - \left(\frac{45}{16}n^3 - \cdots\right)\sin 2\sigma \sin 4\sigma$$
$$+\left(\frac{35}{16}n^4 - \cdots\right)\sin 2\sigma \sin 6\sigma + \left(\frac{225}{256}n^4 - \cdots\right)\sin^2 4\sigma - \cdots$$

and expressing powers and products of trigonometric functions as multiple angles using $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$ and $\sin A \sin B = \frac{1}{2}\left\{\cos(A-B) - \cos(A+B)\right\}$ gives, after some simplification

$$\{F(\sigma)\}^2 = \left(\frac{9}{8}n^2 - \frac{207}{512}n^4 - \cdots\right) - \left(\frac{45}{32}n^3 + \cdots\right)\cos 2\sigma - \left(\frac{9}{8}n^2 - \frac{31}{16}n^4 + \cdots\right)\cos 4\sigma$$
$$+\left(\frac{45}{32}n^3 - \cdots\right)\cos 6\sigma - \left(\frac{785}{512}n^4 + \cdots\right)\cos 8\sigma + \cdots$$

Differentiating with respect to σ and then dividing by 2 gives the 3rd term in equation (237) as

$$\frac{1}{2}\frac{d}{d\sigma}\left[\{F(\sigma)\}^2\right] = \left(\frac{45}{32}n^3 - \cdots\right)\sin 2\sigma + \left(\frac{9}{4}n^2 - \frac{31}{8}n^4 - \cdots\right)\sin 4\sigma$$
$$-\left(\frac{135}{32}n^3 - \cdots\right)\sin 6\sigma + \left(\frac{785}{128}n^4 - \cdots\right)\sin 8\sigma + \cdots \tag{239}$$

Using similar methods the 4th and 5th terms in equation (237) are

$$\frac{1}{6}\frac{d^2}{d\sigma^2}\left[\{F(\sigma)\}^3\right] = -\left(\frac{27}{16}n^3 - \cdots\right)\sin 2\sigma + \left(\frac{135}{16}n^4 - \cdots\right)\sin 4\sigma$$
$$+\left(\frac{81}{16}n^3 - \cdots\right)\sin 6\sigma - \left(\frac{135}{8}n^4 - \cdots\right)\sin 8\sigma + \cdots \tag{240}$$

$$\frac{1}{24}\frac{d^3}{d\sigma^3}\left[\{F(\sigma)\}^4\right] = -\left(\frac{27}{4}n^4 - \cdots\right)\sin 4\sigma + \left(\frac{27}{2}n^4 - \cdots\right)\sin 8\sigma + \cdots \tag{241}$$

Substituting equations (238) to (241) into equation (237) and simplifying gives an equation for ϕ as a function of σ and powers of n as (Lauf 1983, p. 38, eq'n 3.72)

$$\phi = \sigma + \left(\frac{3}{2}n - \frac{27}{32}n^3 - \cdots\right)\sin 2\sigma + \left(\frac{21}{16}n^2 - \frac{55}{32}n^4 + \cdots\right)\sin 4\sigma$$
$$+\left(\frac{151}{96}n^3 + \cdots\right)\sin 6\sigma + \left(\frac{1097}{512}n^4 - \cdots\right)\sin 8\sigma - \cdots \tag{242}$$

where $\sigma = \dfrac{m}{G}$ radians and G is given by equation (229). This very useful series now gives a direct way of computing the latitude given a meridian distance.

Latitude from *Helmert's formula* using Newton-Raphson iteration

In the preceding section, *Helmert's formula* was "reversed" using *Lagrange's theorem* to give equation (242), a direct solution for the latitude ϕ given the meridian distance m and the ellipsoid parameters. As an alternative, a value for ϕ can be computed using the Newton-Raphson method for the real roots of the equation $f(\phi) = 0$ given in the form of an iterative equation

$$\phi_{n+1} = \phi_n - \frac{f(\phi_n)}{f'(\phi_n)} \tag{243}$$

where n denotes the n^{th} iteration and $f(\phi)$ can be obtained from *Helmert's formula* [equation (224)] as

$$f(\phi) = a(1-n)(1-n^2)\left\{b_0\phi - b_2 \sin 2\phi + b_4 \sin 4\phi - b_6 \sin 6\phi + b_8 \sin 8\phi\right\} - m \tag{244}$$

The derivative $f'(\phi) = \dfrac{d}{d\phi}\left\{f(\phi)\right\}$ is given by

$$f'(\phi) = a(1-n)(1-n^2)\left\{b_0 - 2b_2 \cos 2\phi + 4b_4 \cos 4\phi - 6b_6 \cos 6\phi + 8b_8 \cos 8\phi\right\} \tag{245}$$

An initial value for ϕ (for $n = 1$) can be computed from $\phi_1 = \dfrac{m}{a}$ and the functions $f(\phi_1)$ and $f'(\phi_1)$ evaluated from equations (244) and (245) using ϕ_1. ϕ_2 (ϕ for $n = 2$) is now computed from equation (243) and this process repeated to obtain values $\phi_3, \phi_4, \ldots$. This iterative process can be concluded when the difference between ϕ_{n+1} and ϕ_n reaches an acceptably small value.

> Newton-Raphson iteration is a numerical technique used for finding approximations to the roots of real valued functions and is attributed to Isaac Newton (1643-1727) and Joseph Raphson (1648-1715). The technique evolved from investigations into methods of solving cubic and higher-order equations that were of interest to mathematicians in the 17th and 18th centuries. The great French algebraist and statesman François Viète (1540-1603) presented methods for solving equations of second, third and fourth degree. He knew the connection between the positive roots of equations and the coefficients of the different powers of the unknown quantity and it is worth noting that the word

"coefficient" is actually due to Viète. Newton was familiar with Viète's work, and in portions of unpublished notebooks (circa 1664) made extensive notes on Viète's method of solving the equation $x^3 + 30x = 14356197$ and also demonstrated an iterative technique that we would now call the "secant method". In modern notation, this method for solving an equation $f(x) = 0$ is:

$$x_{n+1} = x_n - f(x_n) \Bigg/ \left[\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\right]$$

In Newton's tract of 1669, *De analysi perÆquationes numero terminorum infinitus* ('On analysis by equations unlimited in the number of their terms') – chiefly noted for its initial announcement of the principle of fluxions (the calculus) – is the first recorded discussion of what we may call Newton's iterative method. He applies his method to the solution of the cubic equation $x^3 - 2x - 5 = 0$ and there is no reference to calculus in his development of the method; which suggests that Newton regarded this as a purely algebraic procedure. The process described by Newton required an initial estimate x_0 hence $x = x_0 + p$ where p is a small quantity. This was substituted into the original equation and then expanded using the binomial theorem to give a polynomial in p as

$$\begin{aligned}(x_0 + p)^3 + 2(x_0 + p) - 5 &= 0\\ x_0^3 + 3x_0^2 p + 3x_0 p^2 + p^3 - 2x_0 + 2p - 5 &= 0\\ p^3 + 3x_0 p^2 + (3x_0^2 + 2)\,p &= 5 - 2x_0 - x_0^3\end{aligned}$$

The second and high-order polynomial terms in p were discarded to calculate a numerical approximation p_0 from $3(x_0^2 + 2)\,p_0 = 5 - 2x_0 - x_0^3$. Now $p = p_0 + q$ (q much smaller than p_0) is substituted into the polynomial for p, giving a polynomial in q, and a numerical approximation q_0 calculated by the same manner of discarding second and higher-order terms. This laborious process was repeated until the small numerical terms, calculated at each stage, became insignificant. The final result was the initial estimate x_0 plus the results of the polynomial computations $x = x_0 + p_0 + q_0 + \cdots$ instead of successive estimates x_k being updated and then used in the next computation. This process is significantly different from the iterative technique currently used and known as Newton-Raphson.

In 1690 Joseph Raphson published *Analysis aequationum universalis* in which he presented a new method for solving polynomial equations. As an example, Raphson considers equations of the form $a^3 - ba - c = 0$ in the unknown a and proposes that if g is an estimate of the solution, then a better estimate can be obtained as $g + x$ where

$$x = \frac{c + bg - g^3}{3g^2 - b}$$

Formally, this is of the form $g + x = g - f(g) \big/ f'(g)$ with $f(a) = a^3 - ba - c$. Raphson then applies this formula iteratively to the equation $x^3 - 2x - 5 = 0$. Raphson's formulation was a significant development of Newton's method and the iterative formulation substantially improved the computational convenience. The following comments on Raphson's technique, recorded in the Journal Book of the Royal Society are noteworthy.

> "30 July 1690: Mr Halley related that Mr Ralphson [*sic*] had Invented a method of Solving all sorts of Aquations, and giving their Roots in Infinite Series, which Converge apace, and that he had desired of him an Equation of the fifth power to be proposed to him, to which he return'd Answers true to Seven Figures in much less time than it could have been effected by the Known methods of Vieta."

> "17 December 1690: Mr Ralphson's Book was this day produced by E Halley, wherin he gives a Notable Improvement of ye method of Resolution of all sorts of Equations Shewing, how to Extract their Roots by a General Rule,which doubles the known figures of the Root known by each Operation, So yt by repeating 3 or 4 times he finds them true to Numbers of 8 or 10 places."

It is interesting to note here that Raphson's technique is compared to that of Viète, while Newton's method is not mentioned, although it had, by then, appeared in Wallis' *Algebra.* In the preface to his tract of 1690, Raphson refers to Newton's work but states that his own method is "not only, I believe, not of the same origin, but also, certainly, not with the same development". The two methods were long regarded by users as distinct, but the historian of mathematics, Florian Cajori writing in 1911 recommended the use of the appellation *'Newton-Raphson'* and this is now standard in mathematical texts describing Raphson's method with the notation of calculus.

The information above is drawn from the articles; Thomas, D. J., 1990, 'Joseph Raphson, F.R.S.', *Notes and Records of the Royal Society of London,* Vol. 44, No. 2, (July 1990) pp. 151-167, and Tjalling, J., 1995, 'Historical development of the Newton-Raphson method', *SIAM Review,* Vol. 37, No. 4, pp. 531-551.

1.3.3 Areas on the ellipsoid

The area of a differentially small rectangle on the surface of the ellipsoid is given by equation (184) and (188) as

$$\begin{aligned} dA = J\,d\phi\,d\lambda &= \left(\frac{c}{V^2}\right)^2 \cos\phi\,d\phi\,d\lambda \\ &= \rho\nu\cos\phi\,d\phi\,d\lambda \end{aligned} \tag{246}$$

and integration gives the area of a rectangle bounded by meridians λ_1, λ_2 and parallels ϕ_1, ϕ_2 as

$$A = \int_{\lambda_1}^{\lambda_2}\int_{\phi_1}^{\phi_2} \rho\nu\cos\phi\,d\phi\,d\lambda \tag{247}$$

The area of a zone between parallels ϕ_1, ϕ_2 is

$$\begin{aligned} A &= \int_0^{2\pi}\int_{\phi_1}^{\phi_2} \rho\nu\cos\phi\,d\phi\,d\lambda \\ &= 2\pi\int_{\phi_1}^{\phi_2} \rho\nu\cos\phi\,d\phi \end{aligned} \tag{248}$$

The integral in equation (248) can be evaluated directly (a closed form solution) or by expanding the integrand into a series and then term-by-term integration (a series expansion solution). Both solutions will be developed below.

Series expansion solution for area of zone on ellipsoid

Following Lauf (1983, pp. 38-39), Rapp (1982, pp.41-43) and Baeschlin (1948, pp. 58-62) and using equations (188) we may write the integrand of (248) as

$$\rho\nu\cos\phi = \frac{a^2\left(1-e^2\right)}{W^4}\cos\phi \tag{249}$$

and

$$\frac{1}{W^4} = \frac{1}{\left(1-e^2\sin^2\phi\right)^2} = \left(1-e^2\sin^2\phi\right)^{-2} \tag{250}$$

Using a special case of the binomial series [equation (199) with $\beta = -2$]

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

we have

$$\frac{1}{W^4} = 1 + 2e^2\sin^2\phi + 3e^4\sin^4\phi + 4e^6\sin^6\phi + 5e^8\sin^8\phi + 6e^{10}\sin^{10}\phi + \cdots$$

and hence

$$A = 2\pi a^2\left(1-e^2\right)\int_{\phi_1}^{\phi_2}\left\{\cos\phi + 2e^2\cos\phi\sin^2\phi + 3e^4\cos\phi\sin^4\phi\right.$$
$$\left. + 4e^6\cos\phi\sin^6\phi + 5e^8\cos\phi\sin^8\phi + 6e^{10}\cos\phi\sin^{10}\phi + \cdots\right\}d\phi$$

Using the standard integral $\int \cos x\sin^n x\,dx = \dfrac{\sin^{n+1}x}{n+1}$, term-by-term integration gives

$$A = 2\pi a^2\left(1-e^2\right)\left[\sin\phi + \frac{2e^2}{3}\sin^3\phi + \frac{3e^4}{5}\sin^5\phi + \frac{4e^6}{7}\sin^7\phi + \cdots\right]_{\phi_1}^{\phi_2} \tag{251}$$

To simplify this expression the powers of $\sin\phi$ can be expressed in terms of multiple angles using the standard form

$$\sin^{2n-1}\phi = \frac{(-1)^{n-1}}{2^{2n-2}}\left\{\sin(2n-1)\phi - \binom{2n-1}{1}\sin(2n-3)\phi + \binom{2n-1}{2}\sin(2n-5)\phi\right.$$
$$\left. - \binom{2n-1}{3}\sin(2n-7)\phi + \cdots(-1)^{n-1}\binom{2n-1}{n-1}\sin\phi\right\} \tag{252}$$

Using equation (252) and the binomial coefficients $B_m^{2n-1} = \binom{2n-1}{m}$ computed using equation (202) gives

$$\sin^3\phi = \frac{3}{4}\sin\phi - \frac{1}{4}\sin 3\phi$$
$$\sin^5\phi = \frac{5}{8}\sin\phi - \frac{5}{8}\sin 3\phi + \frac{1}{16}\sin 5\phi$$
$$\sin^7\phi = \frac{35}{64}\sin\phi - \frac{35}{64}\sin 3\phi + \frac{21}{64}\sin 5\phi - \frac{1}{64}\sin 7\phi$$

Substituting these results into equation (251) gives

$$A = 2\pi a^2\left(1-e^2\right)\left[\sin\phi + \frac{2e^2}{3}\left(\frac{3}{4}\sin\phi - \frac{1}{4}\sin 3\phi\right)\right.$$
$$+\frac{3e^4}{5}\left(\frac{5}{8}\sin\phi - \frac{5}{16}\sin 3\phi + \frac{1}{16}\sin 5\phi\right)$$
$$\left.+\frac{4e^6}{7}\left(\frac{35}{64}\sin\phi - \frac{21}{64}\sin 3\phi + \frac{7}{64}\sin 5\phi - \frac{1}{64}\sin 7\phi\right)\right]_{\phi_1}^{\phi_2}$$

Gathering coefficients of $\sin n\phi$ gives

$$\begin{aligned} C_1 &= 1 + \frac{1}{2}e^2 + \frac{3}{8}e^4 + \frac{5}{16}e^6 + \cdots \\ C_3 &= -\left(\frac{1}{6}e^2 + \frac{3}{16}e^4 + \frac{3}{16}e^6 + \cdots\right) \\ C_5 &= \frac{3}{80}e^4 + \frac{1}{16}e^6 + \cdots \\ C_7 &= -\left(\frac{1}{112}e^6 + \cdots\right) \end{aligned} \tag{253}$$

and so the area can be expressed as

$$A = 2\pi a^2\left(1-e^2\right)\left[C_1\sin\phi + C_3\sin 3\phi + C_5\sin 5\phi + C_7\sin 7\phi + \cdots\right]_{\phi_1}^{\phi_2} \tag{254}$$

Evaluating equation (254) with the terminals ϕ_1 and ϕ_2 gives

$$A = 2\pi a^2\left(1-e^2\right)\left\{C_1\left(\sin\phi_2 - \sin\phi_1\right) + C_3\left(\sin 3\phi_2 - \sin 3\phi_1\right)\right.$$
$$\left.+C_5\left(\sin 5\phi_2 - \sin 5\phi_1\right) + C_7\left(\sin 7\phi_2 - \sin 7\phi_1\right) + \cdots\right\}$$

Using the trigonometric relationship $\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$ and with the mean latitude ϕ_m and latitude difference $\Delta\phi$ as

$$\phi_m = \frac{\phi_2 + \phi_1}{2} \quad \text{and} \quad \Delta\phi = \phi_2 - \phi_1 \tag{255}$$

the <u>surface area of a zone on an ellipsoid</u> can be written as

$$A = 4\pi a^2\left(1-e^2\right)\left\{C_1\sin\left(\frac{\Delta\phi}{2}\right)\cos\phi_m + C_3\sin 3\left(\frac{\Delta\phi}{2}\right)\cos 3\phi_m\right.$$
$$\left.+C_5\sin 5\left(\frac{\Delta\phi}{2}\right)\cos 5\phi_m + C_7\sin 7\left(\frac{\Delta\phi}{2}\right)\cos 7\phi_m + \cdots\right\} \tag{256}$$

For square-metre precision on the ellipsoid the number of terms in the series (256) may need to be increased with coefficients (253) extended to higher orders of eccentricity-squared. With the aid of the Computer Algebra System *Maxima* the coefficients (253) have been extended to order e^{12}

$$\begin{aligned}
C_1 &= 1 + \frac{1}{2}e^2 + \frac{3}{8}e^4 + \frac{5}{16}e^6 + \frac{35}{128}e^8 + \frac{63}{256}e^{10} + \frac{231}{1024}e^{12} + \cdots \\
C_3 &= -\left(\frac{1}{6}e^2 + \frac{3}{16}e^4 + \frac{3}{16}e^6 + \frac{35}{192}e^8 + \frac{45}{256}e^{10} + \frac{693}{4096}e^{12} + \cdots\right) \\
C_5 &= \frac{3}{80}e^4 + \frac{1}{16}e^6 + \frac{5}{64}e^8 + \frac{45}{512}e^{10} + \frac{385}{4096}e^{12} + \cdots \\
C_7 &= -\left(\frac{1}{112}e^6 + \frac{5}{256}e^8 + \frac{15}{512}e^{10} + \frac{77}{2048}e^{12} + \cdots\right) \\
C_9 &= \frac{5}{2304}e^8 + \frac{3}{512}e^{10} + \frac{21}{2048}e^{12} + \cdots \\
C_{11} &= -\left(\frac{3}{5632}e^{10} + \frac{7}{4096}e^{12} + \cdots\right) \\
C_{13} &= \frac{7}{53248}e^{12} + \cdots
\end{aligned} \tag{257}$$

[Note: Baeschlin (1948, p. 61) has $\frac{3}{8}e^4$ in term C_3; it should be $\frac{3}{16}e^4$]

Alternatively, using equations (246), (217) and (218) we may write

$$\left(\frac{c}{V^2}\right)^2 = \frac{a^2\left(1-n^2\right)^2}{\left(1+n^2+2n\cos 2\phi\right)^2}$$

and the area of a zone between parallels ϕ_1 and ϕ_2 is

$$A = 2\pi a^2 \int_{\phi_1}^{\phi_2} \frac{\left(1-n^2\right)^2 \cos\phi}{\left(1+n^2+2n\cos 2\phi\right)^2}\, d\phi \tag{258}$$

With the aid of *Maxima* the integral can be expressed as

$$A = 2\pi a^2 \left[D_1 \sin\phi + D_3 \sin 3\phi + D_5 \sin 5\phi + D_7 \sin 7\phi + \cdots\right]_{\phi_1}^{\phi_2} \tag{259}$$

and the surface area of a zone on the ellipsoid is

$$A = 4\pi a^2 \left\{ D_1 \sin\left(\frac{\Delta\phi}{2}\right)\cos\phi_m + D_3 \sin 3\left(\frac{\Delta\phi}{2}\right)\cos 3\phi_m \right.$$
$$\left. + D_5 \sin 5\left(\frac{\Delta\phi}{2}\right)\cos 5\phi_m + D_7 \sin 7\left(\frac{\Delta\phi}{2}\right)\cos 7\phi_m + \cdots \right\} \qquad (260)$$

where the coefficients $\left\{D_k\right\}$ are to order n^6

$$\begin{aligned}
D_1 &= 1 - 2n + n^2 - 2n^3 + 2n^4 - 2n^5 + 2n^6 + \cdots \\
D_3 &= -\frac{2}{3}n + n^2 - \frac{2}{3}n^3 + \frac{2}{3}n^4 - \frac{2}{3}n^5 + \frac{2}{3}n^6 - \cdots \\
D_5 &= \frac{3}{5}n^2 - \frac{4}{5}n^3 + \frac{2}{5}n^4 - \frac{2}{5}n^5 + \frac{2}{5}n^6 - \cdots \\
D_7 &= -\frac{4}{7}n^3 + \frac{5}{7}n^4 - \frac{2}{7}n^5 + \frac{2}{7}n^6 - \cdots \\
D_9 &= \frac{5}{9}n^4 - \frac{2}{3}n^5 + \frac{2}{9}n^6 - \cdots \\
D_{11} &= -\frac{6}{11}n^5 + \frac{7}{11}n^6 - \cdots \\
D_{13} &= \frac{7}{13}n^6 - \cdots
\end{aligned} \qquad (261)$$

[Note: Lauf (1983, p. 79) has $2n^2$ in term D_1; it should be n^2]

Since $e^{12} \approx 9.0$ e-014 and $n^6 \approx 2.2$e-017 the series (260) with the coefficients $\left\{D_k\right\}$ is more 'efficient' than series (256) with coefficients $\left\{C_k\right\}$ since fewer terms are required in the coefficients.

<u>*Closed form solution for area of zone on ellipsoid*</u>

Again following Lauf (1983, pp. 38-39), Rapp (1982, pp.41-43) and Baeschlin (1948, pp. 58-62) and using equations (248), (249) and (250), the area of a latitude zone on the ellipsoid between parallels ϕ_1, ϕ_2 is

$$A = 2\pi a^2 \left(1 - e^2\right) \int_{\phi_1}^{\phi_2} \frac{\cos\phi}{\left(1 - e^2 \sin^2\phi\right)^2} d\phi \qquad (262)$$

For convenience, we consider a special case of (262) between the equator and latitude β and noting that $b^2 = a^2(1 - e^2)$ we write the area of a zone on the ellipsoid between the equator and latitude β as

$$A_\beta = 2\pi b^2 \int_0^\beta \frac{\cos\phi}{\left(1 - e^2 \sin^2\phi\right)^2} d\phi \tag{263}$$

Let $x = \sin\phi$ then $dx = \cos\phi\, d\phi$ and with the terminals $\phi = 0, \beta$ transformed to $x = 0, \sin\beta$ the area becomes

$$A_\beta = 2\pi b^2 \int_0^{\sin\beta} \frac{dx}{\left(1 - e^2 x^2\right)^2} = 2\pi b^2\, I \tag{264}$$

where I is the integral

$$I = \int_0^{\sin\beta} \frac{dx}{\left(1 - e^2 x^2\right)^2} = \frac{1}{e} \int_0^{\sin\beta} \frac{e\, dx}{\left(1 - e^2 x^2\right)^2}$$

To evaluate the integral, let $u = ex$ then $du = e\, dx$ and with the terminals $x = 0, \sin\beta$ transformed to $u = 0, e\sin\beta$ the integral I becomes

$$I = \frac{1}{e} \int_0^{e\sin\beta} \frac{du}{\left(1 - u^2\right)^2}$$

Using the standard result $\int \frac{du}{\left(1 - u^2\right)^2} = \frac{1}{4} \ln\left(\frac{1+u}{1-u}\right) + \frac{1}{2}\left(\frac{u}{1-u^2}\right)$ gives the integral I as

$$I = \frac{1}{2e}\left\{\frac{1}{2}\ln\left(\frac{1 + e\sin\beta}{1 - e\sin\beta}\right) + \frac{e\sin\beta}{1 - e^2\sin^2\beta}\right\} \tag{265}$$

Also, using the inverse hyperbolic function $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $-1 < x < 1$ we may write

$$I = \frac{1}{2e}\left\{\tanh^{-1}\left(e\sin\beta\right) + \frac{e\sin\beta}{1 - e^2\sin^2\beta}\right\} \tag{266}$$

Substituting (265) or (266) into (264) gives the area of a zone on the ellipsoid between the equator and latitude β as

$$A_\beta = \pi b^2 \left\{\frac{1}{2e}\ln\left(\frac{1 + e\sin\beta}{1 - e\sin\beta}\right) + \frac{\sin\beta}{1 - e^2\sin^2\beta}\right\} = \pi b^2 \left\{\frac{\tanh^{-1}\left(e\sin\beta\right)}{e} + \frac{\sin\beta}{1 - e^2\sin^2\beta}\right\} \tag{267}$$

The area of a zone on the ellipsoid between parallels ϕ_1, ϕ_2 is obtained from (267) by replacing β with ϕ_2 and ϕ_1 successively to give

$$A = \pi b^2 \left\{ \frac{1}{2e} \ln\left(\frac{\left(1 + e\sin\phi_2\right)\left(1 - e\sin\phi_1\right)}{\left(1 - e\sin\phi_2\right)\left(1 + e\sin\phi_1\right)} \right) + \frac{\sin\phi_2}{1 - e^2\sin^2\phi_2} - \frac{\sin\phi_1}{1 - e^2\sin^2\phi_1} \right\} \tag{268}$$

1.3.4 Surface area of ellipsoid

The surface area of the ellipsoid can be determined by considering another special case of (262) between the equator $\phi_1 = 0$ and the pole $\phi_2 = \frac{1}{2}\pi$ and noting that $b^2 = a^2(1 - e^2)$.

We write the area of the ellipsoid as

$$A = 4\pi b^2 \int_0^{\frac{\pi}{2}} \frac{\cos\phi}{\left(1 - e^2\sin^2\phi\right)^2} \, d\phi \tag{269}$$

Following the previous method of evaluating the integral leads to

$$A = \frac{2\pi b^2}{e} \left[\frac{1}{2} \ln\left(\frac{1 + e\sin\beta}{1 - e\sin\beta} \right) + \frac{e\sin\beta}{1 - e^2\sin^2\beta} \right]_{\beta=0}^{\beta=\frac{1}{2}\pi}$$

which gives the surface area of the ellipsoid as (Rapp, 1982)

$$A = 2\pi b^2 \left\{ \frac{1}{2e} \ln\left(\frac{1 + e}{1 - e} \right) + \frac{1}{1 - e^2} \right\} \tag{270}$$

Baeschlin (1948) and Lauf (1983) have the equivalent formula

$$A = 2\pi a^2 \left\{ 1 + \frac{1 - e^2}{2e} \ln\left(\frac{1 + e}{1 - e} \right) \right\} \tag{271}$$

1.3.5 Volume of ellipsoid

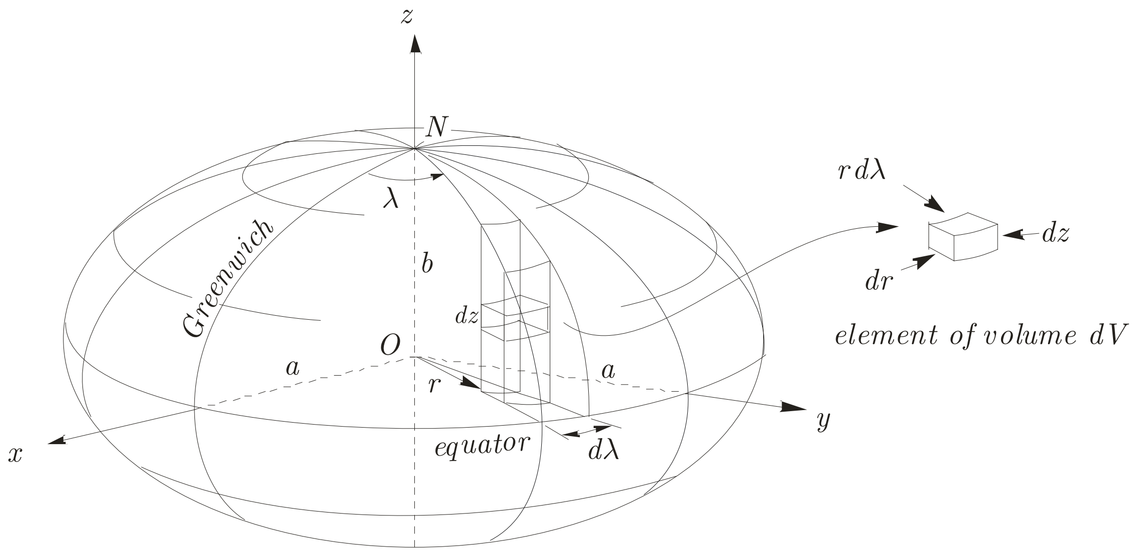


Figure 23: Element of volume $dV = r\,d\lambda\,dr\,dz$

With $r^2 = x^2 + y^2$ the Cartesian equation of the ellipsoid is $\dfrac{r^2}{a^2} + \dfrac{z^2}{b^2} = 1$ and the volume of the ellipsoid is twice the volume of the hemi-ellipsoid or $V_{\text{ellipsoid}} = 2V_{\text{hemi}}$ where

$$V_{\text{hemi}} = \iiint dV = \int_{r_1}^{r_2} \int_{\lambda_1}^{\lambda_2} \int_{z_1(r)}^{z_2(r)} dz\,d\lambda\,r\,dr = \int_{r_1}^{r_2} \left\{ \int_{\lambda_1}^{\lambda_2} \left[\int_{z_1(r)}^{z_2(r)} dz \right] d\lambda \right\} r\,dr$$

and $dV = r\,d\lambda\,dr\,dz$ is the element of volume. The integration terminals are:

(i) for z, from $z_1 = 0$ to $z_2 = \dfrac{b}{a}\sqrt{a^2 - r^2}$

(ii) for λ, from $\lambda_1 = 0$ to $\lambda_2 = 2\pi$

(iii) for r, from $r_1 = 0$ to $r_2 = a$

Evaluating the integrals in turn gives

$$\int_{z_1}^{z_2} dz = [z]_{z_1}^{z_2} = z_2 - z_1 = \frac{b}{a}\sqrt{a^2 - r^2}$$

$$\int_{\lambda_1}^{\lambda_2} \frac{b}{a}\sqrt{a^2 - r^2}\; d\lambda = 2\pi \frac{b}{a}\sqrt{a^2 - r^2}$$

and the volume of a hemi-ellipsoid is

$$V_{\text{hemi}} = \int_{r_1}^{r_2} 2\pi \frac{b}{a}\sqrt{a^2 - x^2}\; r\,dr = 2\pi \frac{b}{a}\int_0^a r\left(a^2 - r^2\right)^{\frac{1}{2}}\; dr$$
$$= -\pi \frac{b}{a}\int_0^a -2r\left(a^2 - r^2\right)^{\frac{1}{2}}\; dr$$

Let $u = a^2 - r^2$, then $du = -2r\,dr$ and the terminals $r = 0, a$ become $u = a^2, 0$ and

$$V_{\text{hemi}} = -\frac{\pi b}{a}\int_{a^2}^{0} u^{\frac{1}{2}}\; du = -\frac{\pi b}{a}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{a^2}^{0} = \frac{2\pi}{3}a^2 b$$

giving the volume of the ellipsoid as

$$V_{\text{ellipsoid}} = 2V_{\text{hemi}} = \frac{4\pi}{3}a^2 b \tag{272}$$

1.3.6 Sphere versus ellipsoid

For certain purposes, it is sufficient to replace the ellipsoid by a sphere of appropriate radius. There are several different spheres (of different radius) that may be adopted.

1. A sphere having a radius equal to the mean of the 3 semi-axes of the ellipsoid

$$R_m = \frac{2a + b}{3} \tag{273}$$

[Note here that in mathematics an ellipsoid is a surface defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and has three semi-axes *a, b,* and *c.* The term *tri-axial* could be used to distinguish this ellipsoid from the (bi-axial) ellipsoid of geodesy $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$]

2. A sphere having the same surface area as the ellipsoid [see equation **Error! Reference source not found.**]

$$\text{surface area of sphere} = \text{surface area of ellipsoid}$$
$$4\pi R_A^2 = 2\pi a^2\left\{1 + \frac{1 - e^2}{2e}\ln\left(\frac{1 + e}{1 - e}\right)\right\}$$

giving

$$R_A^2 = \frac{a^2}{2}\left\{1 + \frac{1-e^2}{2e}\ln\left(\frac{1+e}{1-e}\right)\right\} \tag{274}$$

3. A sphere having the same volume as the ellipsoid [see equation (272)]

$$\text{volume of sphere} = \text{volume of ellipsoid}$$

$$\frac{4}{3}\pi R_V^3 = \frac{4}{3}\pi a^2 b$$

giving

$$R_A^3 = a^2 b \tag{275}$$

4. A sphere having the same quadrant distance as the ellipsoid [see equation (228)]

$$\text{quadrant distance of sphere} = \text{quadrant distance of ellipsoid}$$

$$\frac{2\pi R_Q}{4} = a(1-n)\left(1-n^2\right)\left(1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots\right)\frac{\pi}{2}$$

giving

$$R_Q = a(1-n)\left(1-n^2\right)\left(1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots\right) \tag{276}$$

1.3.7 Geometric parameters of certain ellipsoids

Prior to 1967 the geometric parameters of various ellipsoids were determined from analysis of arc measurements and or astronomic observations in various regions of the Earth, the resulting parameters reflecting the size and shape of "best fit" ellipsoids for those regions – the *International Ellipsoid* of 1924 was adopted by the International Association of Geodesy (at its general assembly in Madrid in 1924) as a best fit of the entire Earth. In 1967 the *International Astronomic Union* (IAU) and the *International Union of Geodesy and Geophysics* (IUGG) defined a set of four physical parameters for the *Geodetic Reference System 1967* based on the theory of a geocentric equipotential ellipsoid. These were: *a,* the equatorial radius of the Earth, *GM,* the geocentric gravitational constant (the product of the Universal Gravitational Constant *G* and the mass of the Earth *M,* including the atmosphere), J_2, the dynamical form factor of the Earth and ω, the angular velocity of the Earth's rotation. The geometric parameters e^2 and f of an ellipsoid (known as the normal ellipsoid) can be derived from these defining parameters as well as the gravitational potential of the ellipsoid and the value of gravity on the ellipsoid (known as normal gravity).

Table 1 shows the geometric parameters of various ellipsoids.

Date	Name	a (metres)	$1/f$
1830	Airy	6377563.396	299.324964600
1830	Everest (India)	6377276.345	300.801700000
1880	Clarke	6378249.145	293.465000000
1924	International	6378388 (exact)	297.0 (exact)
1966	Australian National Spheroid (ANS)	6378160 (exact)	298.25 (exact)
1967	Geodetic Reference System (GRS67)	6378160 (exact)	298.247167427
1980	Geodetic Reference System (GRS80)	6378137 (exact)	298.257222101
1984	World Geodetic System (WGS84)	6378137 (exact)	298.257223563

Table 1: Geometric parameters of selected ellipsoids.
From Appendix A1, Technical Report, Department of Defense World Geodetic System 1984 (NIMA 2000)

The *Geodetic Reference System 1980* (GRS80), adopted by the XVII General Assembly of the IUGG in Canberra, December 1979 is the current estimate with $a = 6378137$ m, $GM = 3986005 \times 10^{8}$ m^{3}s^{-2}, $J_2 = 108263 \times 10^{-8}$ and $\omega = 7292115 \times 10^{-11}$ rad s^{-1} (BG 1988). The *World Geodetic System 1984* (WGS84), the datum for the Global Positioning System (GPS), is based on the GRS80, except that the dynamical form factor of the Earth is expressed in a modified form, causing very small differences between derived parameters of the GRS80 and WGS84 ellipsoids (NIMA 2000). These differences can be regarded as negligible for all practical purposes (e.g., a difference of 0.0001 m in the semi-minor axes). The *Geocentric Datum of Australia* (GDA) uses the GRS80 ellipsoid as its reference ellipsoid.

1.3.8 Constants of the GRS80 ellipsoid

The GRS80 ellipsoid is defined by the two geometric parameters:

semi-major axis: $a = 6378137$ metres
flattening: $f = 1/298.257222101$

These two defining parameters and other computed constants for the GRS80 ellipsoid are:

$$
\begin{aligned}
a &= & &6\,378\,137 \text{ metres}\\
b &= a(1-f) &&= 6\,356\,752.314 \text{ metres}\\
c &= \frac{a}{1-f} &&= 6\,399\,593.626 \text{ metres}\\
e^2 &= f(2-f) &&= 6.694\,380\,023\text{e-003}\\
e'^2 &= \frac{f(2-f)}{(1-f)^2} &&= 6.739\,496\,775\text{e-003}\\
f &= 1/298.257222101 &&= 3.352\,810\,681\text{e-003}\\
n &= \frac{f}{2-f} &&= 1.679\,220\,395\text{e-003}
\end{aligned}
$$

quadrant distance $Q = 10\,001\,965.729$ metres
surface area $A = 5.10065622\text{e+014}$ square-metres
volume $V = 1.08320732\text{e+021}$ cubic-metres

The powers of several of these constants are

$e^2 =$	6.694 380 023	e-003	$e'^2 =$	6.739 496 775	e-003
$e^4 =$	4.481 472 4	e-005	$e'^4 =$	4.542 081 7	e-005
$e^6 =$	3.000 07	e-007	$e'^6 =$	3.06113	e-007
$e^8 =$	2.008	e-009	$e'^8 =$	2.063	e-009
$e^{10} =$	1.3	e-011	$e'^{10} =$	1.4	e-011
$e^{12} =$	9.0	e-014	$e'^{10} =$	9.4	e-014

$f =$	3.352 810 681	e-003	$n =$	1.679 220 395	e-003
$f^2 =$	1.124 133 9	e-005	$n^2 =$	2.819 7811	e-006
$f^3 =$	3.769 0	e-008	$n^3 =$	4.735 0	e-009
$f^4 =$	1.26	e-010	$n^4 =$	7.95	e-012
$f^5 =$	4.2	e-013	$n^5 =$	1.3	e-014
$f^6 =$	1.4	e-015	$n^6 =$	2.2	e-017

From the above, it is clear that the higher powers of n are much smaller than the higher powers of the other constants, so that in general, a series involving powers of n will converge more rapidly than a series involving the powers of other constants. See equations for meridian distance for an example.

The radii of equivalent spheres are:

$$\begin{aligned} R_m &= 6\,371\,008.771 \text{ metres} \\ R_A &= 6\,371\,007.181 \text{ metres} \\ R_V &= 6\,371\,000.790 \text{ metres} \\ R_Q &= 6\,367\,449.146 \text{ metres} \end{aligned}$$

1.3.9 Constants of the GRS80 ellipsoid at latitude φ

At a point P on the surface of the ellipsoid having latitude $\phi = -37°48'33.1234''$ the latitude functions W and V are:

$$\begin{aligned} W^2 &= 1 - e^2 \sin^2 \phi &&= 0.997\,484\,181 \\ W & &&= 0.998\,741\,298 \\ V^2 &= 1 + e'^2 \cos^2 \phi &&= 1.004\,206\,722 \\ V & &&= 1.002\,101\,154 \end{aligned}$$

and the radii of curvature are:

$$\begin{aligned} &\text{prime vertical} & \nu &= \frac{a}{W} = \frac{c}{V} &&= 6\,386\,175.289 \text{ metres} \\ &\text{meridian} & \rho &= \frac{a\left(1-e^2\right)}{W^3} = \frac{c}{V^3} &&= 6\,359\,422.962 \text{ metres} \\ &\text{mean} & \rho_m &= \sqrt{\rho\nu} = \frac{a\left(1-f\right)}{W^2} = \frac{c}{V^2} &&= 6\,372\,785.088 \text{ metres} \end{aligned}$$

The meridian distance from the equator to P is [using equation (224)] with the coefficients b_0, b_2, etc.

$$
\begin{aligned}
b_0 &= 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots && = 1.000\,006\,345\text{e+}000 \\
b_2 &= \frac{3}{2}n + \frac{45}{16}n^3 + \cdots && = 2.518\,843\,909\text{e-}003 \\
b_4 &= \frac{1}{2}\left(\frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots\right) && = 2.643\,557\,858\text{e-}006 \\
b_6 &= \frac{1}{3}\left(\frac{35}{16}n^3 + \cdots\right) && = 3.452\,628\,950\text{e-}009 \\
b_8 &= \frac{1}{4}\left(\frac{315}{128}n^4 + \cdots\right) && = 4.891\,830\,424\text{e-}012
\end{aligned}
$$

gives the meridian distance on the GRS80 ellipsoid as

$$
\begin{aligned}
m &= a(1-n)\left(1-n^2\right)\left\{b_0\phi - b_2\sin 2\phi + b_4\sin 4\phi - b_6\sin 6\phi + b_8\sin 8\phi - \cdots\right\} \\
&= 111132.952547\phi^\circ - 16038.508741\sin(2\phi) \\
&\qquad + 16.832613\sin(4\phi) \\
&\qquad - 0.021984\sin(6\phi) \\
&\qquad + 0.000031\sin(8\phi) \\
&\qquad - \cdots
\end{aligned}
$$

Substituting in the latitude $\phi = -37^\circ 48' \, 33.1234'' = -37.809200944^\circ$

gives $m = 4\,186\,320.340$ metres .

2 TRANSFORMATIONS BETWEEN CARTESIAN COORDINATES *x,y,z* AND GEODETIC COORDINATES

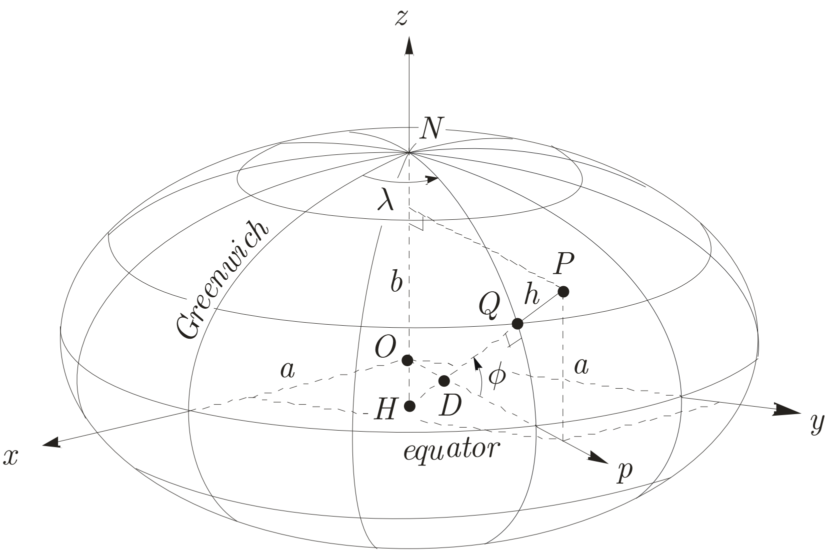


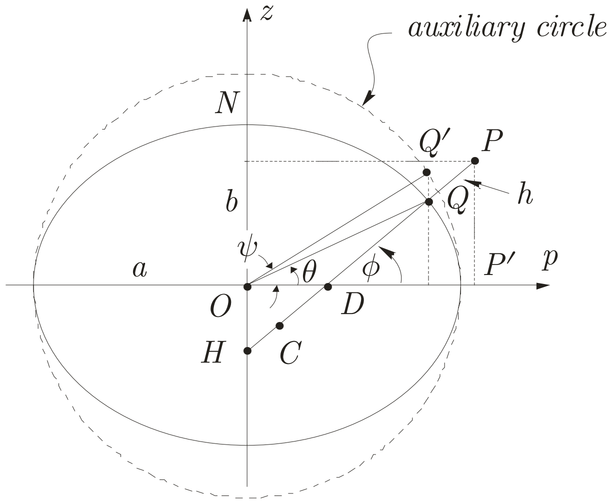
Figure 24: P related to the reference ellipsoid

A point P in space is connected to the ellipsoid via a normal to the surface PH that intersects the surface at Q. H is the intersection of the normal and the z-axis and P is at a height $h = QP$ above/below the ellipsoid. The normal passing through P intersects the equatorial plane xOy at D making an angle ϕ (latitude) and the normal also lies in the meridian plane ONQ, making an angle λ (longitude) with the Greenwich meridian plane. We say then that P has <u>geodetic coordinates</u> ϕ, λ, h. [Note that Q has the same ϕ, λ coordinates as P, but $h = 0$. Q is the projection of P onto the surface via the normal.] P also has <u>Cartesian coordinates</u> x,y,z. The origin of both coordinate systems is at the centre O of the ellipsoid.

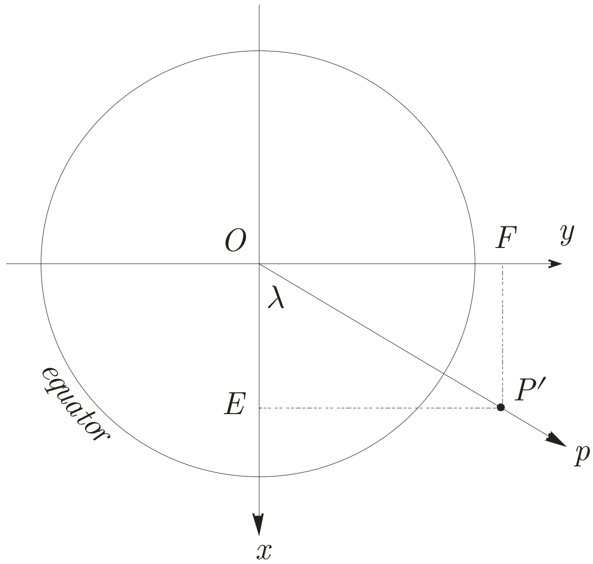
Figure 25 (a) shows the meridian ellipse (meridian normal section) in the meridian plane zOp. P is in this plane and is connected to the ellipse via the normal. The normal cuts the ellipse at Q, the equatorial plane at D and the z-axis at H. The centre of curvature of the meridian ellipse at Q lies on the normal at C. The centre of curvature of the prime vertical normal section (an ellipse in a plane perpendicular to the meridian plane) at Q lies on the normal at H. P' is the projection of P onto the equatorial plane of the ellipsoid.

Also, the auxiliary circle is shown with Q' projected onto the circle via a normal to the p-axis, and the parametric latitude ψ and geocentric latitude θ of Q are shown.

Figure 25(b) shows the equatorial plane of the ellipsoid and P' is the projection of P.



(a) P related to the meridian ellipse



(b) P' on the equatorial plane

Figure 25: Connection between Cartesian and geodetic coordinates

The following relationships have been established; see equations (48) to (51), the interrelationships between ellipse parameters, equations (28) to (37) and the relationships between latitudes, equations (54) to (56).

$$QH = \frac{a}{W} = \frac{c}{V} = \nu \quad \text{radius of curvature in prime vertical section}$$

$$QD = \frac{a}{W}\left(1-e^2\right) = \frac{c}{V}\left(1-e^2\right) = \nu\left(1-e^2\right) = \frac{b}{V}$$

$$OH = \frac{a}{W}e^2 \sin\phi = \frac{c}{V}e^2 \sin\phi = \nu e^2 \sin\phi$$

$$DH = \frac{a}{W}e^2 = \frac{c}{V}e^2 = \nu e^2$$

latitudes: $QDP' = \phi$ (geodetic) $QOP' = \theta$ (geocentric) $Q'OP' = \psi$ (parametric)

latitudes are related by: $\tan\theta = (1-f)^2 \tan\phi$ and $\tan\psi = (1-f)\tan\phi$

Now in Figure 25, with $PH = QH + QP = \nu + h$, we have;

$$OP' = (\nu + h)\cos\phi$$

$$PP' = (\nu + h)\sin\phi - \nu e^2 \sin\phi = \left\{\nu\left(1 - e^2\right) + h\right\}\sin\phi$$

$$OE = OP'\cos\lambda = (\nu + h)\cos\phi\cos\lambda$$

$$OF = OP'\sin\lambda = (\nu + h)\cos\phi\sin\lambda$$

The distances OE, OF and PP' are the x,y,z Cartesian coordinates of P respectively.

For various purposes, it is often required to transform from ϕ,λ,h to x,y,z coordinates and conversely. These transformations are explained in the following sections.

2.1 Cartesian coords *x,y,z* given geodetic coords

The transformation $\phi,\lambda,h \rightarrow x,y,z$ is accomplished by the following:

$$\begin{aligned}
x &= \left(\frac{a}{W} + h\right)\cos\phi\cos\lambda &= \left(\frac{c}{V} + h\right)\cos\phi\cos\lambda &= (\nu + h)\cos\phi\cos\lambda \\
y &= \left(\frac{a}{W} + h\right)\cos\phi\sin\lambda &= \left(\frac{c}{V} + h\right)\cos\phi\cos\lambda &= (\nu + h)\cos\phi\sin\lambda \\
z &= \left(\frac{a\left(1 - e^2\right)}{W} + h\right)\sin\phi &= \left(\frac{b}{V} + h\right)\sin\phi &= \left\{\nu\left(1 - e^2\right) + h\right\}\sin\phi
\end{aligned} \tag{277}$$

with variable domains: $\nu \geq 0,\ \nu + h \geq 0,\ \nu\left(1 - e^2\right) + h \geq 0$ and $-\frac{1}{2}\pi \leq \phi \leq \frac{1}{2}\pi$,

$-\pi < \lambda \leq \pi$

where

$$\begin{aligned}
\nu &= \frac{a}{W} = \frac{c}{V} & \nu\left(1 - e^2\right) &= \frac{b}{V} \\
V^2 &= 1 + e'^2\cos^2\phi & W^2 &= 1 - e^2\sin^2\phi \\
b &= a(1 - f) & c &= \frac{a}{1 - f} \\
e^2 &= f(2 - f) & e'^2 &= \frac{f(2 - f)}{(1 - f)^2}
\end{aligned} \tag{278}$$

2.2 Geodetic coords ϑ given Cartesian coords *x,y,z*

The following relationships can be established from equations (277)

$$
\begin{aligned}
\frac{x}{\cos\lambda} &= \frac{y}{\sin\lambda} = p, \quad \text{provided } \lambda \neq 0, \pm\tfrac{1}{2}\pi, \pi \\
\frac{z}{\sin\phi} &= \frac{p}{\cos\phi} - \nu e^2, \quad \text{provided } \phi \neq 0, \pm\tfrac{1}{2}\pi \\
h &= \frac{p}{\cos\phi} - \nu, \quad \text{provided } \phi \neq \pm\tfrac{1}{2}\pi
\end{aligned}
\tag{279}
$$

where p is the perpendicular distance from the z-axis (the rotational axis)

$$
p = \sqrt{x^2 + y^2} \geq 0 \tag{280}
$$

From the first equation of (279), $\sin\lambda = \dfrac{y}{p}$ and $\cos\lambda = \dfrac{x}{p}$. These relationships seem preferable to evaluate λ than [6] $\tan\lambda = \dfrac{y}{x}$, since each function, sine and cosine, is 2π-periodic whereas tangent is π-periodic.

Choosing $\sin\lambda = \dfrac{y}{p}$ and solving for λ gives

$$
\lambda = \begin{cases}
\left.\begin{array}{l} 0,\ \text{if } x > 0,\ y = 0 \\ \arcsin\dfrac{y}{p},\ \text{if } x > 0,\ y \neq 0 \end{array}\right\} \arcsin\dfrac{y}{p},\ \text{if } x > 0 \\
\left.\begin{array}{l} \pi - \arcsin\dfrac{y}{p},\ \text{if } x < 0,\ y > 0 \\ \pi,\ \text{if } x < 0,\ y = 0 \end{array}\right\} \pi - \arcsin\dfrac{y}{p},\ \text{if } x < 0,\ y \geq 0 \\
-\pi - \arcsin\dfrac{y}{p},\ \text{if } x < 0,\ y < 0
\end{cases}
$$

thus $\lambda \in (-\pi, \pi]$ as desired.

[6] $\tan\lambda = y/x$ is the usual formula for evaluating λ, but requires inspection of x to avoid division by zero, and then inspection of the signs of x and y to determine the correct value $-\pi < \lambda \leq \pi$ since $-\tfrac{1}{2}\pi < \arctan y/x < \tfrac{1}{2}\pi$. Note: the *atan2*$(y, x)$ function, common to many computer languages, will return $-\pi < \lambda \leq \pi$.

These expressions for λ can be combined into

$$\lambda = \tfrac{1}{2}\pi\left(1-\operatorname{sgn}(x)\right)\operatorname{sgn}(y) + \operatorname{sgn}(x)\arcsin\frac{y}{p} \tag{281}$$

using the <u>signum function</u>

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases} \tag{282}$$

Choosing to evaluate λ from $\cos\lambda = \dfrac{x}{p}$ and with similar reasoning to the development of λ from $\sin\lambda = \dfrac{y}{p}$ leads to

$$\lambda = \operatorname{sgn}(y)\arccos\frac{x}{p} \tag{283}$$

The second of equations (279) can be written as

$$p\tan\phi = z + \nu e^2 \sin\phi \tag{284}$$

or

$$z\cos\phi\sqrt{1-e^2\sin^2\phi} + ae^2\sin\phi\cos\phi = p\sin\phi\sqrt{1-e^2\sin^2\phi} \tag{285}$$

using $\nu = \dfrac{a}{\sqrt{1-e^2\sin^2\phi}}$.

The following conditions can be determined.

If $z = 0$, equation (284) reduces to $\sin\phi\left(p - \nu e^2\cos\phi\right) = 0$ whose only feasible solution (provided that $p > \nu e^2\cos\phi$) is $\phi = 0$, since $-\frac{1}{2}\pi \leq \phi \leq \frac{1}{2}\pi$. So, provided $p > \nu e^2\cos\phi$, $\phi \neq 0$ implies $z \neq 0$.

If $x = y = 0$ and $z > 0$, then $p = 0$ and equation (285) reduces to $\cos\phi\left(z + \nu e^2\sin\phi\right) = 0$ whose only feasible solution (provided $z > -\nu e^2\sin\phi$) is $\phi = \frac{1}{2}\pi$ whereas if $z < 0$, the only feasible solution is $\phi = -\frac{1}{2}\pi$, provided $z < -\nu e^2\sin\phi$. So, provided $|z| > \left|\nu e^2\sin\phi\right|$, $\phi \neq \pm\frac{1}{2}\pi$ implies $x \neq y \neq 0$.

We can see from these conditions that there is a spherical region of radius νe^2 centred at *O*, the centre of the ellipsoid, within which, solutions for ϕ will not be unique. <u>We can avoid possible ambiguities in solutions for</u> ϕ <u>by specifying a lower bound for the height</u> h <u>of points as</u> $h \geq -5000$ m.

Using equations (281), (283), (284) and the last of equations (279), the transformation $x, y, z \to \phi, \lambda, h$ is accomplished by

$$\begin{aligned} \lambda &= \begin{cases} \tfrac{1}{2}\pi\left(1-\operatorname{sgn}(x)\right)\operatorname{sgn}(y) + \operatorname{sgn}(x)\arcsin\dfrac{y}{p} \\ \operatorname{sgn}(y)\arccos\dfrac{x}{p} \end{cases} \\ p\tan\phi &= z + \nu e^2 \sin\phi \\ h &= \frac{p}{\cos\phi} - \nu \end{aligned} \tag{286}$$

This transformation is not straightforward. While λ can be readily computed from y and p, or x and p the same cannot be said for ϕ, (and thus h) as there are no simple relationships linking ϕ with x,y,z [see the second of equations (286) where functions of ϕ are on both sides of the equation]. Various techniques for computing ϕ are well documented in the literature and they fall into two categories: iterative solutions and direct solutions.

Direct solutions for ϕ involve the formation of quartic equations (an algebraic equation of the 4^{th}-degree) which are reduced to cubic equations having a single real root [e.g., Paul 1973, Ozone 1985, Lapaine 1990 and Vermeille 2002].

Iterative solutions are generally simpler than direct solutions and some do not require a 2^{nd} iteration if points are reasonably close to the ellipsoid $\left(-5{,}000\,\text{m} \le h \le 10{,}000\,\text{m}\right)$, and they fall into two groups; (i) trigonometric [e.g., Bowring 1973, Borkowski 1989, Laskowski 1991 and Jones 2002] where formula involve trigonometric relationships and (ii) vector [e.g., Lin and Wang 1995 and Pollard 2002] where formula are derived from vector relationships. Iterative solutions are usually easier to program and generally require less evaluations of square-roots, exponentiations and trigonometric functions than direct solutions. These evaluations are computer-time-intensive and slow the computation of multiple positions; hence iterative solutions for ϕ are often regarded as being faster than direct solutions.

We will discuss five methods: (i) Successive Substitution, (ii) Newton-Raphson Iteration (iii) Bowring's method, (iv) Lin & Wang's method and (v) Paul's method as being representative of the various methods of computing ϕ and performing the transformation $x, y, z \to \phi, \lambda, h$.

2.2.1 Successive Substitution

This technique, described in various forms in the literature, owes its popularity to its programming simplicity. The basis for the method is the second of equations (286), namely

$$p \tan \phi = z + \nu e^2 \sin \phi \tag{287}$$

An approximate value ϕ_0 is used in the right-hand-side (RHS) of equation (287) to evaluate $p \tan \phi_1$ (and hence ϕ_1) on the left-hand-side (LHS). This new value, ϕ_1 is then used in the RHS to give the next value, $p \tan \phi_2$ (and hence ϕ_2). This procedure is repeated until the difference between successive LHS values ϕ_n, ϕ_{n+1} reaches an acceptable limit. Thus the iteration, providing certain conditions are met (see below), converges to a solution for $p \tan \phi$ and hence ϕ.

The starting value ϕ_0 is obtained from the relationship between geocentric and geodetic latitude: $(1-f)^2 \tan \phi_0 = \tan \theta \simeq \dfrac{z}{p}$ giving

$$\tan \phi_0 = \frac{z}{p(1-f)^2} \tag{288}$$

A note on convergence criteria of Successive Substitution is appropriate here.

Equation (287) can be expressed as

$$f(\phi) = z + \nu e^2 \sin \phi - p \tan \phi$$

and the sufficient condition for Successive Substitution to converge to a solution for ϕ is $\left| f'(\phi) < 1 \right|$ where $f'(\phi) = \dfrac{d}{d\phi} f(\phi)$. The evaluation of this derivative is covered in the next section where it is found to be $f'(\phi) = \dfrac{c}{V^3} e'^2 \cos \phi - \dfrac{p}{\cos^2 \phi}$. Now the extreme values for $f'(\phi)$ are when $\phi = 0$ and $\phi = \pm\left(\dfrac{\pi}{2} - \varepsilon\right)$ where ε is a very small quantity, since this method is applicable only if $p > 0$ and:

(i) when $\phi = 0$ and $\cos \phi = 1$, $V^3 = \left(1 + e'^2\right)^{\frac{3}{2}} \simeq 1$ and $f'(\phi) \simeq ce'^2 - p$;

(ii) when $\phi = \pm\left(\dfrac{\pi}{2} - \varepsilon\right)$ and $\cos \phi \simeq \varepsilon, V^3 \simeq 1$ and $f'(\phi) \simeq ce'^2 \varepsilon - \dfrac{p}{\varepsilon}$

So, provided that $p > ce'^2$ the criteria for convergence should be satisfied. For the GRS80 ellipsoid $ce'^2 \simeq 43130$ m.

2.2.2 Newton-Raphson Iteration

The rate of convergence for ϕ in the Successive Substitution solution can be improved by using Newton-Raphson iteration for the real roots of the equation $f(\phi) = 0$ given in the form of an iterative equation

$$\phi_{n+1} = \phi_n - \frac{f(\phi_n)}{f'(\phi_n)} \tag{289}$$

where n denotes the n^{th} iteration and $f(\phi)$, from equations (286) is

$$f(\phi) = z + \nu e^2 \sin\phi - p\tan\phi \tag{290}$$

and the derivative $f'(\phi) = \dfrac{d}{d\phi}\{f(\phi)\}$ is given by

$$f'(\phi) = \nu e^2 \cos\phi + e^2 \sin\phi \frac{d\nu}{d\phi} - \frac{p}{\cos^2\phi} \tag{291}$$

Now $\nu = \dfrac{c}{V}$ and noting that $V^2 = 1 + e'^2 \cos^2\phi$ and $\dfrac{dV}{d\phi} = -\dfrac{e'^2}{V}\cos\phi\sin\phi$ then

$$\frac{d\nu}{d\phi} = -\frac{c}{V^2}\frac{dV}{d\phi} = \frac{c}{V^3}e'^2\cos\phi\sin\phi \tag{292}$$

Substituting equation (292) into equation (291) gives

$$\begin{aligned} f'(\phi) &= e^2\cos\phi\left\{\frac{c}{V} + \frac{ce'^2\sin^2\phi}{V^3}\right\} - \frac{p}{\cos^2\phi} \\ &= \frac{c}{V^3}e^2\cos\phi\left(V^2 + e'^2 - e'^2\cos^2\phi\right) - \frac{p}{\cos^2\phi} \\ &= \frac{c}{V^3}e^2\cos\phi\left(V^2 + e'^2 + 1 - V^2\right) - \frac{p}{\cos^2\phi} \\ &= \frac{c}{V^3}e^2\cos\phi\left(1 + e'^2\right) - \frac{p}{\cos^2\phi} \end{aligned} \tag{293}$$

But, from equation (32) $1 + e'^2 = \dfrac{e'^2}{e^2}$ and equation (293) becomes

$$f'(\phi) = \frac{c}{V^3}e'^2\cos\phi - \frac{p}{\cos^2\phi} \tag{294}$$

Alternatively, from equations (28) and (25) $1+e'^2=\dfrac{a^2}{b^2}=\dfrac{c^2}{a^2}$, and equation (293) becomes

$$f'(\phi)=\frac{c^3}{a^2V^3}e^2\cos\phi-\frac{p}{\cos^2\phi} \tag{295}$$

A starting value ϕ_0 can be obtained from equation (288) and the iteration continued until the correction factor $\Delta\phi_n=\left|\dfrac{f(\phi_n)}{f'(\phi_n)}\right|$ in equation (289) reaches an acceptably small magnitude.

2.2.3 Bowring's method

This method (Bowring 1976) is one of the iterative methods, but second or third iterations are not required if we are dealing with Earth-bound points where $-5{,}000\,\text{m}\leq h\leq 10{,}000\,\text{m}$ (Bowring 1976, p.326).

From the parametric equations of the evolute of an ellipse [see equations (78)] we may write the z- and p-coordinates of the centre of curvature C in Figures 25(a) and 26 as

$$\begin{aligned} z_C &= -be'^2\sin^3\psi \\ p_C &= ae^2\cos^3\psi \end{aligned} \tag{296}$$

and the expression for $\tan\phi$ in terms of the parametric latitude ψ can be obtained from Figure 26 as

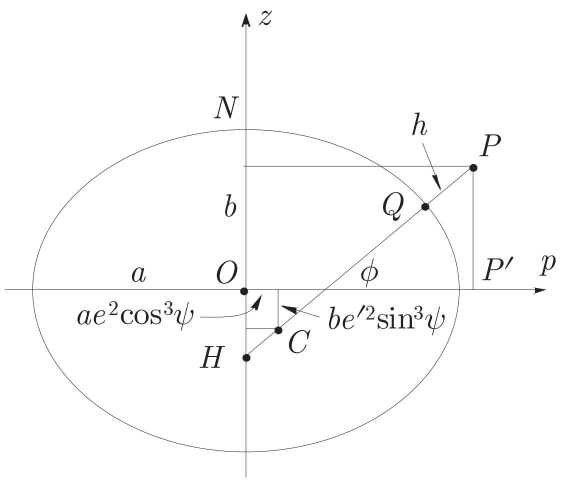


Figure 26: Coordinates of centre of curvature C

$$\tan\phi = \frac{z + be'^2 \sin^3\psi}{p - ae^2\cos^3\psi} \tag{297}$$

Equation (297) in conjunction with $\tan\psi = (1-f)\tan\phi$ can be solved simultaneously for ϕ, and if necessary ψ, iteratively.

Similarly to Successive Substitution, a starting value ψ_0 is obtained from the relationship between geocentric and parametric latitude: $(1-f)\tan\psi_0 = \tan\theta \simeq \frac{z}{p}$ giving

$$\tan\psi_0 = \frac{z}{p(1-f)} \tag{298}$$

Bowring (1976) shows that for all Earth-bound points $\left(-5,000\,\text{m} \le h \le 10,000\,\text{m}\right)$ the maximum error in ϕ, induced by using only a single iteration, is $0.000\,000\,030''$.

For points in space where $h > 10,000$ m a second (or perhaps even a third) iteration of Bowring's equation may be required. These iterations may be performed in two ways:

(i) Successive Substitution, i.e., calculating a new value of the parametric latitude ψ from $\tan\psi = (1-f)\tan\phi$ and using this new value in the RHS of equation (297) to give an improved vale of ϕ*;* and so on until the difference between successive LHS values ϕ_n, ϕ_{n+1} reaches an acceptable limit.

(ii) Using $\tan\psi = \left(1-f\right)\tan\phi$ in (297) and re-arranging gives

$$\frac{\tan\psi}{1-f} = \frac{z + be'^2\sin^3\psi}{p - ae^2\cos^3\psi} \tag{299}$$

and ψ obtained by Newton-Raphson iteration for the real roots of the equation $f\left(\psi\right) = 0$ given in the form of an iterative equation

$$\psi_{n+1} = \psi_n - \frac{f\left(\psi_n\right)}{f'\left(\psi_n\right)} \tag{300}$$

where n denotes the n^{th} iteration and $f\left(\psi\right)$, from equation (299) is

$$f\left(\psi\right) = \left(p - ae^2\cos^3\psi\right)\tan\psi - \left(1-f\right)\left(z + be'^2\sin^3\psi\right) \tag{301}$$

and the derivative $f'(\phi) = \frac{d}{d\phi}\left\{f(\phi)\right\}$. Noting that $\frac{d}{d\psi}\tan\psi = \sec^2\psi$ and

$$\frac{d}{d\psi}\cos^3\psi = -3\cos^2\psi\sin\psi;\ \frac{d}{d\psi}\sin^3\psi = 3\sin^2\psi\cos\psi \text{ then}$$

$$\begin{aligned} f'\left(\psi\right) &= \left(p - ae^2\cos^3\psi\right)\sec^2\psi + \tan\psi\left(3ae^2\cos^2\psi\sin\psi\right) - \left(1-f\right)3be'^2\sin^2\psi\cos\psi \\ &= p\sec^2\psi - ae^2\cos\psi + 3ae^2\cos\psi\sin^2\psi - 3\left(1-f\right)be'^2\cos\psi\sin^2\psi \end{aligned}$$

Now from the interrelationship between ellipse parameters (see section 1.1.5) we find $\left(1-f\right)be'^2 = ae^2$ and the derivative can be expressed as

$$f'\left(\psi\right) = \frac{p}{\cos^2\psi} - ae^2\cos\psi \tag{302}$$

A starting value ψ_0 can be obtained from equation (298) and the iteration continued until the continued until the correction factor $\Delta\psi_n = \left|\frac{f\left(\psi_n\right)}{f'\left(\psi_n\right)}\right|$ in equation (300) reaches an acceptably small magnitude.

It should be noted here that for h = 26,000,000 m (A GPS satellite has an approximate orbital height of 20,000,000 m) only two iterations are required for acceptable accuracy.

2.2.4 Lin and Wang's method

This elegant iterative method (Lin & Wang 1995) uses Newton-Raphson iteration to evaluate a scalar multiplier q of the normal vector to the ellipsoid. Once q is obtained, simple relationships between Cartesian coordinates of P, and its normal projection Q onto the ellipsoid at P, are used to evaluate the Cartesian coordinates of Q; thus allowing geodetic latitude ϕ to be computed from the relationship $\frac{z_Q}{\sqrt{x_Q^2 + y_Q^2}} = (1-f)^2\tan\phi$, since $\tan\theta \simeq \frac{z_Q}{p_Q} = \frac{z_Q}{\sqrt{x_Q^2 + y_Q^2}}$.

An outline of the development of the necessary equations is given below, and relies upon some aspects of differential geometry of surfaces.

Recalling section 1.2.1 Differential Geometry of Space Curves, a curve C in space is defined as the locus of the terminal points P of a position vector $\mathbf{r}(t)$ defined by a single scalar parameter t,

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{303}$$

and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are fixed unit Cartesian vectors in the directions of the x,y,z coordinate axes.

If this curve C is constrained to lie on a surface

$$\varphi = \varphi(x, y, z) = \text{ constant} \tag{304}$$

then the scalar components of the position vector (303) must satisfy equation (304); thus

$$\varphi\big(x(t), y(t), z(t)\big) = \text{ constant} \tag{305}$$

As the parameter t varies, the terminal point of the vector sweeps out C on the surface and if s is the arc length from some point on C, then s is a function of t and x,y,z are functions of s. The unit tangent vector $\hat{\mathbf{t}}$ of the curve C in the direction of increasing s is given by

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} \tag{306}$$

and at a fixed point P on the surface $\varphi(x, y, z) = \text{ constant}$ there are an infinite number of curves C passing through P. Hence, the tangent vectors at P will all lie in the tangent plane of the surface at P and the normal to this plane is the surface normal vector of φ at P. Differentiating equation (305) with respect to s, we obtain via the chain rule

$$\frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\varphi}{\partial y}\frac{dy}{ds} + \frac{\partial\varphi}{\partial z}\frac{dz}{ds} = 0 \tag{307}$$

Now, define a vector differential operator ∇ (*del* or *nabla*) as

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{308}$$

so that

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k} \tag{309}$$

Then equation (307) can be expressed as

$$\nabla\varphi \bullet \hat{\mathbf{t}} = 0 \tag{310}$$

Because this scalar product is zero, $\nabla\varphi$ must be perpendicular to the tangent plane and so in the direction of the surface normal. Lin & Wang use this result in the following way.

The Cartesian equation of the ellipsoid at Q is

$$\frac{x_Q^2}{a^2}+\frac{y_Q^2}{a^2}+\frac{z_Q^2}{b^2}=1 \tag{311}$$

and this is a surface having the general form $\varphi\left(x_Q, y_Q, z_Q\right)=$ constant. Using equation (309) then

$$\nabla\varphi\big|_Q=\frac{2x_Q}{a^2}\mathbf{i}+\frac{2y_Q}{a^2}\mathbf{j}+\frac{2z_Q}{b^2}\mathbf{k} \tag{312}$$

The vector $\overrightarrow{QP}$ is

$$\overrightarrow{QP}=\left(x_P-x_Q\right)\mathbf{i}+\left(y_P-y_Q\right)\mathbf{j}+\left(z_P-z_Q\right)\mathbf{k} \tag{313}$$

and $\overrightarrow{QP}$ (being normal to the surface at Q) is also a scalar (denoted by q) multiple of the surface normal $\nabla\varphi$ at Q, thus

$$\overrightarrow{QP}=\frac{2qx_Q}{a^2}\mathbf{i}+\frac{2qy_Q}{a^2}\mathbf{j}+\frac{2qz_Q}{b^2}\mathbf{k} \tag{314}$$

Equating the scalar components of vectors (313) and (314), and re-arranging gives

$$\frac{x_Q}{a}=\frac{ax_P}{a^2+2q};\quad \frac{y_Q}{a}=\frac{ay_P}{a^2+2q};\quad \frac{z_Q}{b}=\frac{bz_P}{b^2+2q} \tag{315}$$

Squaring each equation above, then substituting into equation (311) gives

$$f(q)=\frac{a^2p_P^2}{\left(a^2+2q\right)^2}+\frac{b^2z_P^2}{\left(b^2+2q\right)^2}=1 \tag{316}$$

and q can be obtained by using Newton-Raphson iteration for the real roots of the equation $f(q)=0$ given in the form of an iterative equation

$$q_{n+1}=q_n-\frac{f\left(q_n\right)}{f'\left(q_n\right)} \tag{317}$$

where n denotes the n^{th} iteration, and the function $f\left(q_n\right)$ and its derivative $f'\left(q_n\right)=\dfrac{d}{dq_n}\left\{f\left(q_n\right)\right\}$ are given as

$$f\left(q_n\right)=\frac{a^2p_P^2}{\left(a^2+2q\right)^2}+\frac{b^2z_P^2}{\left(b^2+2q\right)^2}-1 \tag{318}$$

$$f'(q_n) = -4\left\{\frac{a^2 p_P^2}{\left(a^2 + 2q\right)^3} + \frac{b^2 z_P^2}{\left(b^2 + 2q\right)^3}\right\} \tag{319}$$

Lin and Wang (1995, p. 301) suggest an initial approximation

$$q_0 = \frac{ab\left(a^2 z_P^2 + b^2 p_P^2\right)^{3/2} - a^2 b^2 \left(a^2 z_P^2 + b^2 p_P^2\right)}{2\left(a^4 z_P^2 + b^4 p_P^2\right)} \tag{320}$$

After calculating q, p_Q and z_Q are obtained from equations (315) as

$$\begin{aligned} p_Q &= \frac{a^2 p_P}{a^2 + 2q} \\ z_Q &= \frac{b^2 z_P}{b^2 + 2q} \end{aligned} \tag{321}$$

and the latitude ϕ and height h obtained from

$$\begin{aligned} \tan\phi &= \frac{z_Q}{(1-f)^2 p_Q} = \frac{z_Q}{\left(1-e^2\right) p_Q} \\ h &= \pm\sqrt{\left(p_P - p_Q\right)^2 + \left(z_P - z_Q\right)^2} \end{aligned} \tag{322}$$

noting that h is negative if $p_P + |z_P| < p_Q + |z_Q|$

The attraction of this method, compared to other iterative solutions, is that no trigonometric functions are used in the calculation of $\tan\phi$ and h.

It should be noted here that for all Earth-bound points $\left(-5,000\,\text{m} \le h \le 10,000\,\text{m}\right)$ Lin & Wang's method requires only a single iteration for acceptable accuracy.

For points in space where $h > 10,000$ m a second (or perhaps even a third) iteration may be required and for h = 26,000,000 m (A GPS satellite has an approximate orbital height of 20,000,000 m) two iterations are required for acceptable accuracy.

2.2.5 Paul's method

This method (Paul 1973) is direct in so far as $\tan\phi$ is obtained from a simple closed equation, but only after several intermediate variables have been evaluated. An outline of the development of the necessary equations is given below.

From the second of equations (279)

$$p\tan\phi - z = \nu e^2 \sin\phi \tag{323}$$

and the equation for ν (radius of curvature) can be re-arranged as

$$\begin{aligned}\nu = \frac{a}{W} &= \frac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} \\ &= \frac{a}{\left(\cos^2\phi + \sin^2\phi - e^2\sin^2\phi\right)^{\frac{1}{2}}} \\ &= \frac{a}{\left(\cos^2\phi + \left(1-e^2\right)\sin^2\phi\right)^{\frac{1}{2}}}\end{aligned}$$

Substituting this result for ν into equation (323) gives

$$p\tan\phi - z = \frac{ae^2\sin\phi}{\left(\cos^2\phi + \left(1-e^2\right)\sin^2\phi\right)^{\frac{1}{2}}} = \frac{ae^2\tan\phi}{\left(1+\left(1-e^2\right)\tan^2\phi\right)^{\frac{1}{2}}} \tag{324}$$

Squaring both sides of equation (324) and re-arranging gives

$$\left(p^2\tan^2\phi - 2pz\tan\phi + z^2\right)\left(1+\left(1-e^2\right)\tan^2\phi\right) = a^2e^4\tan^2\phi$$

Multiplying the LHS, then gathering terms and re-arranging so that the RHS is zero and then dividing both sides by $1-e^2$ gives

$$p^2\tan^4\phi - 2pz\tan^3\phi + \left(z^2 + \frac{p^2 - a^2e^4}{1-e^2}\right)\tan^2\phi - \frac{2pz}{1-e^2}\tan\phi + \frac{z^2}{1-e^2} = 0 \tag{325}$$

Letting

$$\beta = \frac{p^2 - a^2e^4}{1-e^2} \tag{326}$$

and multiplying equation (325) by p^2 gives

$$p^4\tan^4\phi - 2z\,p^3\tan^3\phi + \left(z^2+\beta\right)p^2\tan^2\phi - \frac{2zp^2}{1-e^2}\,p\tan\phi + \frac{z^2p^2}{1-e^2} = 0 \tag{327}$$

Equation (327) is a quartic equation in $\tan\phi$, for which there are direct solutions for the four possible values of $\tan\phi$; which could all be real, or all complex, or some real and the others complex. It turns out that this equation can be expressed in terms of a subsidiary value ζ which itself a function of the roots of a cubic equation having a single real root. Hence by means of appropriate substitutions we are able to find the single real value of $\tan\phi$ from equation (327). The method of solving quartic equations, by reduction to a cubic whose solution is known, was first developed in 1540 by Lodivico Ferrari (1522-1565) who resided in Bologna, Papal States (now Italy), the technique set out below is a modification of his general method.

Let

$$p\tan\phi = \zeta + \frac{z}{2} \tag{328}$$

where ζ is a subsidiary variable; then expressions for $p^2\tan^2\phi$, $p^3\tan^3\phi$ and $p^4\tan^4\phi$ can be substituted into equation (327) to give, after some algebra, a quartic equation in ζ, but with no ζ^3-term.

$$\zeta^4 + \left(\beta - \frac{z^2}{2}\right)\zeta^2 - \alpha z\zeta + z^2\left(\frac{\beta}{4} - \frac{z^2}{16}\right) = 0 \tag{329}$$

where

$$\alpha = \frac{p^2 + a^2e^4}{1-e^2} \tag{330}$$

Now, assuming a solution of equation (329) is

$$\zeta = \sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3} \tag{331}$$

where t_1, t_2, t_3 are the roots of the cubic equation $t^3 + a_1t^2 + a_2t + a_3 = 0$, the following relationships can be established, noting that $A = t_1 + t_2 + t_3$

$$\begin{aligned}
\zeta^2 &= \left(t_1 + t_2 + t_3\right) + 2\sqrt{t_1t_2} + 2\sqrt{t_1t_3} + 2\sqrt{t_2t_3} \\
&= A + 2\sqrt{t_1t_2} + 2\sqrt{t_1t_3} + 2\sqrt{t_2t_3} \\
\zeta^4 &= A^2 + 4A\sqrt{t_1t_2} + 4A\sqrt{t_1t_3} + 4A\sqrt{t_2t_3} \\
&\quad + 8t_3\sqrt{t_1t_2} + 8t_2\sqrt{t_1t_3} + 8t_1\sqrt{t_2t_3} \\
&\quad + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 \\
2\left(\zeta^2 - A\right) &= 4\sqrt{t_1t_2} + 4\sqrt{t_1t_3} + 4\sqrt{t_2t_3} \\
2A\left(\zeta^2 - A\right) &= 4A\sqrt{t_1t_2} + 4A\sqrt{t_1t_3} + 4A\sqrt{t_2t_3}
\end{aligned} \tag{332}$$

Using the relationships (332) we can write

$$\zeta^4 = A^2 + 2A\left(\zeta^2 - A\right) + 8t_3\sqrt{t_1t_2} + 8t_2\sqrt{t_1t_3} + 8t_1\sqrt{t_2t_3} + 4\left(t_1t_2 + t_1t_3 + t_2t_3\right) \tag{333}$$

and re-arranging equation (333) gives another quartic equation in ζ

$$\zeta^4 - 2\left(t_1 + t_2 + t_3\right)\zeta^2 - 8\sqrt{t_1t_2t_3}\,\zeta + \left\{\left(t_1 + t_2 + t_3\right)^2 - 4\left(t_1t_2 + t_1t_3 + t_2t_3\right)\right\} = 0 \tag{334}$$

Comparing the coefficients of ζ^2 and ζ, and the constant terms in equations (329) and (334) gives the following relationships

$$\begin{aligned} t_1 + t_2 + t_3 &= \frac{z^2}{4} - \frac{\beta}{2} \\ t_1t_2t_3 &= \frac{\alpha^2 z^2}{64} \\ t_1t_2 + t_1t_3 + t_2t_3 &= \frac{\beta^2}{16} - \frac{\beta z^2}{8} \end{aligned}$$

Now t_1, t_2, t_3 are the roots of the cubic equation $t^3 + a_1t^2 + a_2t + a_3 = 0$ and the coefficients are $a_1 = -\left(t_1 + t_2 + t_3\right)$, $a_2 = t_1t_2 + t_1t_3 + t_2t_3$, $a_3 = -t_1t_2t_3$ respectively. Using the sums and products in equation (331) gives a solution for ζ in terms of a single real-root t_1 as

$$\zeta = \sqrt{t_1} + \sqrt{\frac{z^2}{4} - \frac{\beta}{2} - t_1 + \frac{az}{4\sqrt{t_1}}} \tag{335}$$

Substituting expressions for a_1, a_2, a_3 into the general cubic equation gives

$$t^3 + \left(\frac{\beta}{2} - \frac{z^2}{4}\right)t^2 + \left(\frac{\beta^2}{16} - \frac{\beta z^2}{8}\right)t - \frac{\alpha^2 z^2}{64} = 0 \tag{336}$$

A further substitution

$$t = \left(\frac{\beta + z^2}{6}\right)u + \frac{z^2}{12} - \frac{\beta}{6} \tag{337}$$

enables equation (336) to be reduced to a form

$$4u^3 - 3u = q \tag{338}$$

with a single real-root u_1 having the solution

$$u_1 = \frac{1}{2}\left\{\sqrt[3]{q + \sqrt{q^2 - 1}} + \sqrt[3]{q - \sqrt{q^2 - 1}}\right\} \tag{339}$$

where

$$q = 1 + \frac{27z^2\left(\alpha^2 - \beta^2\right)}{2\left(\beta + z^2\right)^3} \tag{340}$$

Thus having x,y,z (hence p) for a point related to an ellipsoid, $\tan\phi$ is obtained by computing the following variables in order: α from equation (330), β from equation (326), q from equation (340), u_1 from equation (339), t_1 from equation (337), ζ from equation (335) noting that all square-roots in equation (335) have the same sign as z, and finally $\tan\phi$ from equation (328).

3 MATLAB FUNCTIONS

MATLAB® (an acronym for MATrix LABoratory) is a powerful computer program designed to perform scientific calculations. It was originally designed to perform matrix mathematics but has evolved into a flexible computing system capable of solving almost any technical problem. MATLAB has its own computer language (often called the MATLAB language) that looks like the C computer language, and an extensive suite of technical functions. MATLAB functions can be easily constructed using the MATLAB *editor* and executed from the MATLAB *command window*. Results from MATLAB functions are easily presented in the command window.

3.1 ELLIPSOID CONSTANTS

The MATLAB function *ellipsoid_1.m* given below, computes constants of an ellipsoid given the defining parameters: semi-major axis a and denominator of flattening *flat* where $f = 1/flat$. The function is designed to be run from the MATLAB command window with output from the function printed in the MATLAB command window.

MATLAB function *ellipsoid_1.m*

```
function ellipsoid_1(a,flat)
%
% ellipsoid_1(a,flat)  Function takes semi-major axis 'a' of ellipsoid
%   and reciprocal of flattening 'flat' and computes some geometric
%   constants of the ellipsoid.  Note that the flattening f = 1/flat.
%   e.g., ellipsoid_1(6378137,298.257222101) will return the constants of
%         the GRS80 ellipsoid

%==========================================================================
% Function:  ellipsoid_1
%
% Useage:    ellipsoid_1(a,flat);
%
% Author:
%  Rod Deakin,
%  School of Mathematical and Geospatial Sciences, RMIT University,
%  GPO Box 2476V, MELBOURNE VIC 3001
%  AUSTRALIA
%  email: rod.deakin@rmit.edu.au
%
% Date:
%  Version 1.0  31 January 2008
%  Version 1.1  25 April   2012
%
% Remarks:
%  Function takes semi-major axis 'a' of ellipsoid and reciprocal of
%  flattening 'flat' and computes some geometric constants of the
%  ellipsoid.  Note that the flattening f = 1/flat.
```

```
%
% References:
%  [1] Deakin, R.E. and Hunter, M.N., 2008, 'Geometric Geodesy', School of
%         Mathematical and Geospatial Sciences, RMIT University, February
%         2008.
%
% Variables:
%  A         - surface area of ellipsoid
%  a         - semi-major axis of ellipsoid
%  b         - semi-minor axis of ellipsoid
%  c         - polar radius of curvature
%  e         - 1st eccentricity of ellipsoid
%  e2        - 1st eccentricity squared
%  e4,e6,e8,
%  e10,e12   - powers of e2
%  ep2       - 2nd eccentricity squared
%  ep4,ep6,ep8,
%  ep10,ep12 - powers of ep2
%  f         - flattening of ellipsoid
%  f2,f3,f4,
%  f5,f6     - powers of flattening
%  flat      - reciprocal of flattening f = 1/flat
%  n         - ellipsoid constant
%  n2,n3,n4,
%  n5,n6     - powers of n
%  Q         - Quadrant distance of ellipsoid
%  Ra        - radius of sphere having same surface area as ellipsoid
%  Rm        - radius of sphere having mean of ellipsoid axes
%  Rv        - radius of sphere having same volume as ellipsoid
%  Rq        - radius of sphere having same quadrant distance as ellipsoid
%  V         - volume of ellipsoid
%==========================================================================

% compute flattening f and powers of f
f = 1/flat;
f2 = f*f;
f3 = f2*f;
f4 = f3*f;
f5 = f4*f;
f6 = f5*f;

% compute semi-minor axis b and polar radius of curvature c
b = a*(1-f);
c = a/(1-f);

% compute eccentricity squared e2, eccentricity e and powers of e2
e2  = f*(2-f);
e   = sqrt(e2);
e4  = e2*e2;
e6  = e4*e2;
e8  = e6*e2;
e10 = e8*e2;
e12 = e10*e2;

% compute 2nd eccentricity squared (e_primed squared) and powers of ep2
ep2  = e2/(1-e2);
ep4  = ep2*ep2;
ep6  = ep4*ep2;
ep8  = ep6*ep2;
ep10 = ep8*ep2;
ep12 = ep10*ep2;

% compute n
n = f/(2-f);
n2 = n*n;
```

```
n3 = n2*n;
n4 = n3*n;
n5 = n4*n;
n6 = n5*n;

% compute quadrant distance Q
Q = a*(1-n)*(1-n2)*(1 + 9/4*n2 + 225/64*n4)*pi/2;

% surface area of ellipsoid
A = 2*pi*a^2*(1 + (1-e2)/(2*sqrt(e2))*log((1+e)/(1-e)));

% volume of ellipsoid
V = 4*pi/3*a*a*b;

% compute radii of equivalent spheres
Rm = (2*a + b)/3;
Ra = sqrt(a^2/2*(1 + (1-e2)/(2*e)*log((1+e)/(1-e))));
Rv = (a*a*b)^(1/3);
Rq = a*(1-n)*(1-n2)*(1 + 9/4*n2 + 225/64*n4);

% print values to screen
fprintf('\n\nEllipsoid Constants:');
fprintf('\n==================');
fprintf('\n          semi-major axis     a = %11.3f metres',a);
fprintf('\n          semi-minor axis     b = %11.3f metres',b);
fprintf('\npolar radius of curvature     c = %11.3f metres',c);
fprintf('\n     eccentricity squared    e2 = %14.9e',e2);
fprintf('\n 2nd eccentricity squared   ep2 = %14.9e',ep2);
fprintf('\n               flattening     f = %14.9e',f);
fprintf('\ndenominator of flattening  flat = %13.9f',flat);
fprintf('\n                              n = %14.9e',n);
fprintf('\n        quadrant distance     Q = %12.3f metres',Q);
fprintf('\n             surface area     A = %.8e square-metres',A);
fprintf('\n                   volume     V = %.8e cubic-metres',V);
fprintf('\n\nradii of equivalent spheres');
fprintf('\n              mean radius    Rm = %12.3f metre',Rm);
fprintf('\n                     area    Ra = %12.3f metres',Ra);
fprintf('\n                   volume    Rv = %12.3f metres',Rv);
fprintf('\n                 quadrant    Rq = %12.3f metres',Rq);
fprintf('\n\npowers of constants');
fprintf('\n e2 = %14.9e      ep2 = %14.9e',e2,ep2);
fprintf('\n e4 = %14.9e      ep4 = %14.9e',e4,ep4);
fprintf('\n e6 = %14.9e      ep6 = %14.9e',e6,ep6);
fprintf('\n e8 = %14.9e      ep8 = %14.9e',e8,ep8);
fprintf('\ne10 = %14.9e     ep10 = %14.9e',e10,ep10);
fprintf('\ne12 = %14.9e     ep12 = %14.9e',e12,ep12);
fprintf('\n\n  f = %14.9e        n = %14.9e',f,n);
fprintf('\n f2 = %14.9e       n2 = %14.9e',f2,n2);
fprintf('\n f3 = %14.9e       n3 = %14.9e',f3,n3);
fprintf('\n f4 = %14.9e       n4 = %14.9e',f4,n4);
fprintf('\n f5 = %14.9e       n5 = %14.9e',f5,n5);
fprintf('\n f6 = %14.9e       n6 = %14.9e',f6,n6);
fprintf('\n\n');
```

Help message for MATLAB function *ellipsoid_ 1.m*

```
>> help ellipsoid_1

  ellipsoid_1(a,flat)  Function takes semi-major axis 'a' of ellipsoid
    and reciprocal of flattening 'flat' and computes some geometric
    constants of the ellipsoid.  Note that the flattening f = 1/flat.
    e.g., ellipsoid_1(6378137,298.257222101) will return the constants of
          the GRS80 ellipsoid
```

Output from MATLAB function *ellipsoid_ 1.m*

```
>> ellipsoid_1(6378137,298.257222101);

Ellipsoid Constants:
===================
          semi-major axis     a = 6378137.000 metres
          semi-minor axis     b = 6356752.314 metres
polar radius of curvature     c = 6399593.626 metres
     eccentricity squared    e2 = 6.694380023e-003
 2nd eccentricity squared   ep2 = 6.739496775e-003
               flattening     f = 3.352810681e-003
denominator of flattening  flat = 298.257222101
                              n = 1.679220395e-003
        quadrant distance     Q = 10001965.729 metres
             surface area     A = 5.10065622e+014 square-metres
                   volume     V = 1.08320732e+021 cubic-metres

radii of equivalent spheres
              mean radius    Rm =  6371008.771 metre
                     area    Ra =  6371007.181 metres
                   volume    Rv =  6371000.790 metres
                 quadrant    Rq =  6367449.146 metres

powers of constants
 e2 = 6.694380023e-003      ep2 = 6.739496775e-003
 e4 = 4.481472389e-005      ep4 = 4.542081679e-005
 e6 = 3.000067923e-007      ep6 = 3.061134483e-007
 e8 = 2.008359477e-009      ep8 = 2.063050598e-009
e10 = 1.344472156e-011     ep10 = 1.390392285e-011
e12 = 9.000407545e-014     ep12 = 9.370544321e-014

  f = 3.352810681e-003        n = 1.679220395e-003
 f2 = 1.124133946e-005       n2 = 2.819781134e-006
 f3 = 3.769008303e-008       n3 = 4.735033988e-009
 f4 = 1.263677129e-010       n4 = 7.951165642e-012
 f5 = 4.236870177e-013       n5 = 1.335175951e-014
 f6 = 1.420542358e-015       n6 = 2.242054687e-017

>>
```

3.2 MERIDIAN DISTANCE

Three MATLAB functions are given below. The first function, *mdist.m,* uses *Helmert's formula* [equation (224)] to compute the meridian distance m given the ellipsoid parameters a (semi-major axis of ellipsoid)and *flat* (the denominator of the flattening f) and the latitude *lat* in the form ddd.mmss, where a latitude of $-37^\circ\, 48'\, 33.1234''$ would be input into the function as -37.48331234. The function is designed to be run from the MATLAB command window with output from the function printed in the MATLAB command window.

The second function, *latitude.m,* computes the latitude ϕ given the meridian distance m and the ellipsoid parameters a (semi-major axis of ellipsoid), *flat* (the denominator of the flattening f). The function uses equation (242), the series formula developed by reversing

Helmert's formula, and is designed to be run from the MATLAB command window with output from the function printed in the MATLAB command window.

The third function, *latitude2.m,* computes the latitude ϕ given the meridian distance m and the ellipsoid parameters a (semi-major axis of ellipsoid), *flat* (the denominator of the flattening f). The function uses the Newton-Raphson iterative scheme to compute the latitude from *Helmert's formula* [equation (224)] and is designed to be run from the MATLAB command window with output from the function printed in the MATLAB command window.

MATLAB function *mdist.m*

```
%
%
function mdist(a,flat,lat)
%
% MDIST(A,FLAT,LAT)  Function computes the meridian distance on an
%  ellipsoid defined by semi-major axis (A) and denominator of flattening
%  (FLAT) from the equator to a point having latitude (LAT) in d.mmss format.
%  For example: mdist(6378137, 298.257222101, -37.48331234) will compute the
%  meridian distance for a point having latitude -37 degrees 48 minutes
%  33.1234 seconds on the GRS80 ellipsoid (a = 6378137, f = 1/298.257222101)

%--------------------------------------------------------------------------
% Function:  mdist()
%
% Usage:     mdist(a,flat,lat)
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  22 March 2006
%
% Purpose:   Function mdist(a,f,lat) will compute the meridian distance on
%            an ellipsoid defined by semi-major axis a and flat, the
%            denominator of the flattening f where f = 1/flat.  Latitude is
%            given in d.mmss format.
%
% Functions required:
%            decdeg = dms2deg(dms)
%            [D,M,S] = DMS(DecDeg)
%
% Variables: a      - semi-major axis of spheroid
%            b0,b1,b2, - coefficients
%            d2r    - degree to radian conversion factor 57.29577951...
%            n,n2,n3, etc - powers of n
%            f      - f = 1/flat is the flattening of ellipsoid
%            flat   - denominator of flattening of ellipsoid
%
% Remarks:   Helmert's formula for meridian distance is given in
%            Lauf, G.B., 1983, Geodesy and Map Projections,
%            TAFE Publications Unit, Collingwood, p. 36, eq'n 3.58.
%            A derivation can also be found in Deakin, R.E., Meridian
%            Distance, Lecture Notes, School of Mathematical and
%            Geospatial Sciences, RMIT University, March 2006.
```

```
%------------------------------------------------------------------------

% degree to radian conversion factor
d2r = 180/pi;

% compute flattening f and ellipsoid constant n
f = 1/flat;
n = f/(2-f);

% powers n
n2 = n*n;
n3 = n2*n;
n4 = n3*n;

% coefficients in Helmert's series expansion for meridian distance
b0  = 1+(9/4)*n2+(225/64)*n4;
b2  = (3/2)*n+(45/16)*n3;
b4  = (1/2)*((15/8)*n2+(105/32)*n4);
b6  = (1/3)*((35/16)*n3);
b8  = (1/4)*((315/128)*n4);

% compute meridian distance
x = abs(dms2deg(lat)/d2r);
term1 = b0*x;
term2 = b2*sin(2*x);
term3 = b4*sin(4*x);
term4 = b6*sin(6*x);
term5 = b8*sin(8*x);

mdist = a*(1-n)*(1-n2)*(term1-term2+term3-term4+term5);

% print result to screen
fprintf('\n a  = %12.4f',a);
fprintf('\n f  = 1/%13.9f',flat);
[D,M,S] = DMS(x*d2r);
if D == 0 && lat < 0
    fprintf('\nLatitude =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude      = %4d %2d %9.6f (D M S)',D,M,S);
end
    fprintf('\nMeridian dist = %15.6f',mdist);

fprintf('\n\n');
```

Help message for MATLAB function *mdist.m*

```
>> help mdist


  MDIST(A,FLAT,LAT)  Function computes the meridian distance on an
   ellipsoid defined by semi-major axis (A) and denominator of flattening
   (FLAT) from the equator to a point having latitude (LAT) in d.mmss format.
   For example: mdist(6378137, 298.257222101, -37.48331234) will compute the
   meridian distance for a point having latitude -37 degrees 48 minutes
   33.1234 seconds on the GRS80 ellipsoid (a = 6378137, f = 1/298.257222101)
```

Output from MATLAB function *mdist.m*

```
>> mdist(6378137,298.257222101,-37.48331234)

 a  = 6378137.0000
 f  = 1/298.257222101
Latitude      =   37 48 33.123400 (D M S)
Meridian dist =  4186320.340377

>>
```

MATLAB function *latitude.m*

```
function latitude(a,flat,mdist)
%
% LATITUDE(A,FLAT,MDIST)  Function computes the latitude of a point
%   on an ellipsoid defined by semi-major axis (A) and denominator of
%   flattening (FLAT) given the meridian distance (MDIST) from the
%   equator to the point.
%   For example: latitude(6378137,298.257222101,5540847.041561) should
%   return a latitude of 50 degrees 00 minutes 00 seconds for a meridian
%   distance of 5540847.041561m on the GRS80 ellipsoid (a = 6378137, f =
%   1/298.257222101)

%----------------------------------------------------------------------------
% Function:  latitude()
%
% Usage:     latitude(a,f,mdist)
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  23 March 2006
%
% Functions required:
%            [D,M,S] = DMS(DecDeg)
%
% Purpose:
%    Function latitude() will compute the latitude of a point on on an
%    ellipsoid defined by semi-major axis (a) and denominator of
%    flattening (flat) given meridian distance (m_dist) from the
%    equator to the point.
%
% Variables:
%    a       - semi-major axis of spheroid
%    d2r     - degree to radian conversion factor 57.29577951...
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    lat     - latitude (degrees)
%    g       - mean length of an arc of one radian of the meridian
%    mdist   - meridian distance
%    n       - eta, n = f/(2-f)
%    n2,n4,  - powers of eta
%    s       - sigma s = m_dist/g
%    s2,s3,  - powers of sigma
%
% Remarks:
%    For an ellipsoid defined by semi-major axis (a) and flattening (f) the
%    meridian distance (mdist) can be computed by series expansion
```

```
%    formulae (see function mdist.m).  The reverse operation, given a
%    meridian distance on a defined ellipsoid to calculate the latitude,
%    can be achieved by series formulae published in THE AUSTRALIAN GEODETIC
%    DATUM Technical Manual Special Publication 10, National Mapping Council
%    of Australia, 1986 (section 4.4, page 24-25).  The development of these
%    formulae are given in Lauf, G.B., 1983, GEODESY AND MAP PROJECTIONS,
%    Tafe Publications, Vic., pp.35-38.
%    This function is generally used to compute the "footpoint latitude"
%    which is the latitude for which the meridian distance is equal to the
%    y-coordinate divided by the central meridian scale factor, i.e.,
%    latitude for m_dist = y/k0.
%---------------------------------------------------------------------------

% degree to radian conversion factor
d2r = 180/pi;

% calculate flatteninf f and ellipsoid constant n and powers of n
f  = 1/flat;
n  = f/(2.0-f);
n2 = n*n;
n3 = n2*n;
n4 = n3*n;

% calculate the mean length an arc of one radian on the meridian
g = a*(1-n)*(1-n2)*(1+9/4*n2+225/64*n4);

% calculate sigma (s) and powers of sigma
s  = mdist/g;
s2 = 2.0*s;
s4 = 4.0*s;
s6 = 6.0*s;
s8 = 8.0*s;

% calculate the latitude (in radians)
lat = s + (3*n/2 - 27/32*n3)*sin(s2)...
        + (21/16*n2 - 55/32*n4)*sin(s4)...
        + (151/96*n3)*sin(s6)...
        + (1097/512*n4)*sin(s8);

% convert latitude to degrees
lat = lat*d2r;

% print result to screen
fprintf('\n a  = %12.4f',a);
fprintf('\n f  = 1/%13.9f',flat);
[D,M,S] = DMS(lat);
if D == 0 && lat < 0
    fprintf('\nLatitude =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude      = %4d %2d %9.6f (D M S)',D,M,S);
end
    fprintf('\nMeridian dist = %15.6f',mdist);

fprintf('\n\n');
```

Help message for MATLAB function *latitude.m*

```
>> help latitude


  LATITUDE(A,FLAT,MDIST)  Function computes the latitude of a point
    on an ellipsoid defined by semi-major axis (A) and denominator of
    flattening (FLAT) given the meridian distance (MDIST) from the
    equator to the point.
    For example: latitude(6378137,298.257222101,5540847.041561) should
    return a latitude of 50 degrees 00 minutes 00 seconds for a meridian
    distance of 5540847.041561m on the GRS80 ellipsoid (a = 6378137, f =
    1/298.257222101)
```

Output from MATLAB function *latitude.m*

```
>> latitude(6378137,298.257222101,4186320.340377)

 a  = 6378137.0000
 f  = 1/298.257222101
Latitude      =   37 48 33.123400 (D M S)
Meridian dist =  4186320.340377

>>
```

MATLAB function *latitude2.m*

```
function latitude2(a,flat,mdist)
%
% LATITUDE2(A,FLAT,MDIST)  Function computes the latitude of a point
%   on an ellipsoid defined by semi-major axis (A) and denominator of
%   flattening (FLAT) given the meridian distance (MDIST) from the
%   equator to the point.
%   For example: latitude(6378137,298.257222101,5540847.041561) should
%   return a latitude of 50 degrees 00 minutes 00 seconds for a meridian
%   distance of 5540847.041561m on the GRS80 ellipsoid (a = 6378137, f =
%   1/298.257222101)

%------------------------------------------------------------------------
% Function:  latitude2()
%
% Usage:     latitude2(a,f,mdist)
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  23 March 2006
%
% Functions required:
%            [D,M,S] = DMS(DecDeg)
%
% Purpose:
%    Function latitude2() will compute the latitude of a point on on an
%    ellipsoid defined by semi-major axis (a) and denominator of
%    flattening (flat) given meridian distance (mdist) from the
%    equator to the point.
```

```
%
% Variables:
%    a       - semi-major axis of spheroid
%    b0,b1,b2,... coefficients in Helmert's formula
%    corrn   - correction term in Newton-Raphson iteration
%    count   - iteration number
%    d2r     - degree to radian conversion factor 57.29577951...
%    F       - a function of latitude (Helmert's formula)
%    Fdash   - the derivative of F
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    lat     - latitude
%    mdist   - meridian distance
%    n       - eta, n = f/(2-f)
%    n2,n4,  - powers of eta
%
% Remarks:
%    For an ellipsoid defined by semi-major axis (a) and flattening (f) the
%    meridian distance (mdist) can be computed by series expansion
%    formulae (see function mdist.m).  The reverse operation, given a
%    meridian distance on a defined ellipsoid to calculate the latitude,
%    can be achieved by using Newton's Iterative scheme.
%--------------------------------------------------------------------------

% degree to radian conversion factor
d2r = 180/pi;

% calculate flattening f and ellipsoid constant n and powers of n
f  = 1/flat;
n  = f/(2.0-f);
n2 = n*n;
n3 = n2*n;
n4 = n3*n;

% coefficients in Helmert's series expansion for meridian distance
b0  = 1+(9/4)*n2+(225/64)*n4;
b2  = (3/2)*n+(45/16)*n3;
b4  = (1/2)*((15/8)*n2+(105/32)*n4);
b6  = (1/3)*((35/16)*n3);
b8  = (1/4)*((315/128)*n4);

% set the first approximation of the latitude and then Newton's iterative
% scheme where F is the function of latitude and Fdash is the derivative of
% the function F
lat   = mdist/a;
corrn = 1;
count = 0;
while (abs(corrn)>1e-10)
    F = a*(1-n)*(1-n2)*(b0*lat...
                        - b2*sin(2*lat)...
                        + b4*sin(4*lat)...
                        - b6*sin(6*lat)...
                        + b8*sin(8*lat)) - mdist;
    Fdash = a*(1-n)*(1-n2)*(b0...
                            - 2*b2*cos(2*lat)...
                            + 4*b4*cos(4*lat)...
                            - 6*b6*cos(6*lat)...
                            + 8*b8*cos(8*lat));
    corrn  = -F/Fdash;
    lat = lat + corrn;
    count = count+1;
end

% convert latitude to degrees
lat = lat*d2r;
```

```
% print result to screen
fprintf('\n a  = %12.4f',a);
fprintf('\n f  = 1/%13.9f',flat);
[D,M,S] = DMS(lat);
if D == 0 && lat < 0
    fprintf('\nLatitude =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude      = %4d %2d %9.6f (D M S)',D,M,S);
end
fprintf('\nMeridian dist = %15.6f',mdist);
fprintf('\niterations    = %d',count);

fprintf('\n\n');
```

Help message for MATLAB function *latitude2.m*

```
>> help latitude2


  LATITUDE2(A,FLAT,MDIST)  Function computes the latitude of a point
    on an ellipsoid defined by semi-major axis (A) and denominator of
    flattening (FLAT) given the meridian distance (MDIST) from the
    equator to the point.
    For example: latitude(6378137,298.257222101,5540847.041561) should
    return a latitude of 50 degrees 00 minutes 00 seconds for a meridian
    distance of 5540847.041561m on the GRS80 ellipsoid (a = 6378137, f =
    1/298.257222101)
```

Output from MATLAB function *latitude2.m*

```
>> latitude2(6378137,298.257222101,4186320.340377)

 a  = 6378137.0000
 f  = 1/298.257222101
Latitude      =   37 48 33.123400 (D M S)
Meridian dist =  4186320.340377
iterations    = 3

>>
```

MATLAB functions *DMS.m* and *dms2deg.m*

MATLAB functions *mdist.m, latitude.m* and *latitude2.m* call functions *DMS.m* and *dms2deg.m* to convert decimal degrees to degrees, minutes and seconds (for printing) and ddd.mmss format to decimal degrees. These functions are shown below.

```
function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;
if(DecDeg<0)
  D = -D;
end
return
```

```
function DecDeg=dms2deg(DMS)
% DMS2DEG
%  Function to convert from DDD.MMSS format to decimal degrees

x = abs(DMS);
D = fix(x);
x = (x-D)*100;
M = fix(x);
S = (x-M)*100;
DecDeg = D + M/60 + S/3600;
if(DMS<0)
  DecDeg = -DecDeg;
end
return
```

3.3 CARTESIAN TO GEODETIC TRANSFORMATION

The MATLAB function *Geo2Cart.m* given below, computes Cartesian coordinates *x,y,z* of a point related to an ellipsoid defined by semi-major axis a and denominator of flattening *flat* given geodetic coordinates ϕ (*lat*), λ (*lon*) and h. ϕ and λ are assumed to be in radians and the function returns the *x,y,z* coordinates in a vector.

MATLAB function *Geo2Cart.m*

```
function [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%
% [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%   Function computes the Cartesian coordinates X,Y,Z of a point
%   related to an ellipsoid defined by semi-major axis (a) and the
%   denominator of the flattening (flat) given geodetic coordinates
%   latitude (lat), longitude (lon) and ellipsoidal height (h).
%   Latitude and longitude are assumed to be in radians.

%--------------------------------------------------------------------------
% Function:  Geo2Cart()
%
% Usage:     [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   6 April    2006
%            Version  1.0  20 August   2007
%            Version  1.1   3 March    2008
%
% Functions required:
%    radii()
%
% Purpose:
%    Function Geo2Cart() will compute Cartesian coordinates X,Y,Z
%    given geodetic coordinates latitude, longitude (both in radians) and
%    height of a  point related to an ellipsoid defined by semi-major axis
%    (a) and denominator of flattening (flat).
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    e2      - 1st-eccentricity squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor axis of ellipsoid
%    rm      - radius of curvature of meridian section of ellipsoid
%    rp      - radius of curvature of prime vertical section of ellipsoid
%
```

```
% References:
% [1] Deakin, R.E. and Hunter, M.N., 2008, GEOMETRIC GEODESY - PART A,
%       School of Mathematical and Geospatial Sciences, RMIT University,
%       Melbourne, AUSTRALIA, March 2008.
% [2] Gerdan, G.P. & Deakin, R.E., 1999, 'Transforming Cartesian
%       coordinates X,Y,Z to geogrpahical coordinates phi,lambda,h', The
%       Australian Surveyor, Vol. 44, No. 1, pp. 55-63, June 1999.
%--------------------------------------------------------------------------

% calculate flattening f and ellipsoid constant e2
f   = 1/flat;
e2  = f*(2-f);

% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);

% compute Cartesian coordinates X,Y,Z
p = (rp+h)*cos(lat);
X = p*cos(lon);
Y = p*sin(lon);
Z = (rp*(1-e2)+h)*sin(lat);
```

Help message for MATLAB function *Geo2Cart.m*

```
>> help Geo2Cart

  [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
    Function computes the Cartesian coordinates X,Y,Z of a point
    related to an ellipsoid defined by semi-major axis (a) and the
    denominator of the flattening (flat) given geodetic coordinates
    latitude (lat), longitude (lon) and ellipsoidal height (h).
    Latitude and longitude are assumed to be in radians.

>>
```

Operation of MATLAB function *Geo2Cart.m*

```
>> d2r = 180/pi;
>> a = 6378137;
>> flat = 298.257222101;
>> lat = -50/d2r;
>> lon = -150/d2r;
>> h = 10000;
>> [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
>> [X,Y,Z]'

ans =

         -3563081.36230554
         -2057145.98367164
         -4870449.48202417

>>
```

3.4 GEODETIC TO CARTESIAN TRANSFORMATION

Five MATLAB functions are given below. All of the functions compute ϕ (*lat*), λ (*lon*) and h of a point related to an ellipsoid defined by semi-major axis (a) and denominator of flattening (*flat*) given Cartesian coordinates *x,y,z*. All of the functions return ϕ, λ, h as a vector (with ϕ and λ in radians) as well as printing output to the MATLAB command window.

The first function: *Cart2Geo_Substitution.m* is an iterative solution using the method of Successive Substitution. The second function: *Cart2Geo_Newton.m* is an iterative solution using Newton-Raphson iteration. The third function: *Cart2Geo_Bowring.m* is an iterative solution using Bowring's method with a single iteration. The fourth function: *Cart2Geo_Lin.m* is an iterative solution using Lin and Wang's method and the fifth function: *Cart2Geo_Paul.m* is a direct solution using Paul's method.

In the the output from each of these functions shown below, *x,y,z* Cartesian coordinates are first computed for a given ϕ, λ, h and ellipsoid using *Geo2Cart.m* and then these *x,y,z* coordinates are used in the function to compute ϕ, λ, h.

MATLAB function *Cart2Geo_Substitution.m*

```
function [lat,lon,h] = Cart2Geo_Substitution(a,flat,X,Y,Z)
%
% [lat,lon,h] = Cart2Geo_Substitution(a,flat,X,Y,Z)
%   Function computes the latitude (lat), longitude (lon) and height (h)
%   of a point related to an ellipsoid defined by semi-major axis (a)
%   and denominator of flattening (flat) given Cartesian coordinates
%   X,Y,Z.  Latitude and longitude are returned as radians.  This function
%   uses Successive Substitution for converting X,Y,Z to lat,lon,height.

%--------------------------------------------------------------------------
% Function:  Cart2Geo_Substitution()
%
% Usage:     [lat,lon,h] = Cart2Geo_Substitution(a,flat,X,Y,Z);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   9 February 2008
%
% Functions required:
%    radii()
%
```

```
% Purpose:
%    Function Cart2Geo_Substitution() will compute latitude, longitude
%    (both in radians) and height of a  point related to an ellipsoid
%    defined by semi-major axis (a) and denominator of flattening (flat)
%    given Cartesian coordinates X,Y,Z.
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    count   - integer counter for number of iterations
%    corrn   - correction to approximate value
%    d2r     - degree to radian conversion factor = 57.29577951...
%    e2      - 1st eccentricity squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor-axis of ellipsoid
%    rm      - radius of curvature of meridian section of ellipsoid
%    rp      - radius of curvature of prime vertical section of ellipsoid
%
% Remarks:
%    This function uses Successive Substitution, see Refences [1] and [2].
%
% References:
% [1] Gerdan, G.P. & Deakin, R.E., 1999, 'Transforming Cartesian
%       coordinates X,Y,Z to geogrpahical coordinates phi,lambda,h', The
%       Australian Surveyor, Vol. 44, No. 1, pp. 55-63, June 1999.
% [2] Deakin, R.E. and Hunter, M.N., 2008, GEOMETRIC GEODESY - PART A,
%       School of Mathematical and Geospatial Sciences, RMIT University,
%       Melbourne, AUSTRALIA, March 2008.
%-------------------------------------------------------------------------

% Set degree to radian conversion factor
d2r = 180/pi;

% calculate flattening f and ellipsoid constant e2
f   = 1/flat;
e2  = f*(2-f);

% compute 1st approximation of geodetic latitude for the Simple Iteration
p   = sqrt(X*X + Y*Y);
lat = atan(Z/(p*(1-e2)));
corrn = 1;
count = 0;
while (abs(corrn)>1e-10)
%   Compute radii of curvature
    [rm,rp] = radii(a,flat,lat);
%   Compute new approximation of latitude
    new_lat = atan((Z+rp*e2*sin(lat))/p);
    corrn = lat-new_lat;
    count = count+1;
    lat = new_lat;
end;

% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);

% compute longitude and height
lon = atan2(Y,X);
h   = p/cos(lat) - rp;

% Print results to screen
fprintf('\n\nCartesian to Geographic - Simple Iteration');
fprintf('\n==========================================');
```

```
fprintf('\nEllipsoid:');
fprintf('\nsemi-major axis a = %13.3f',a);
fprintf('\nflattening      f = 1/%13.9f',flat);
fprintf('\nCartesian coordinates:');
fprintf('\nX = %13.3f',X);
fprintf('\nY = %13.3f',Y);
fprintf('\nZ = %13.3f',Z);
fprintf('\nGeodetic coordinates:');
[D,M,S] = DMS(lat*d2r);
if D == 0 && lat < 0
    fprintf('\nLatitude  =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  = %4d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon*d2r);
if D == 0 && lon < 0
    fprintf('\nLongitude =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude = %4d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\nHeight    = %13.3f',h);
fprintf('\nIterations    = %3d',count);
fprintf('\n\n');
```

Help message for MATLAB function *Cart2Geo_ Substitution.m*

```
>> help Cart2Geo_Substitution

  [lat,lon,h] = Cart2Geo_Substitution(a,flat,X,Y,Z)
    Function computes the latitude (lat), longitude (lon) and height (h)
    of a point related to an ellipsoid defined by semi-major axis (a)
    and denominator of flattening (flat) given Cartesian coordinates
    X,Y,Z.  Latitude and longitude are returned as radians.  This function
    uses Successive Substitution for converting X,Y,Z to lat,lon,height.

>>
```

Output from MATLAB function *Cart2Geo_ Substitution.m*

```
>> d2r = 180/pi;
>> a = 6378137;
>> flat = 298.257222101;
>> lat = -50/d2r;
>> lon = -150/d2r;
>> h = 10000;
>> [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
>> [X,Y,Z]'

ans =

        -3563081.36230554
        -2057145.98367164
        -4870449.48202417

>> [lat,lon,h] = Cart2Geo_Substitution(a,flat,X,Y,Z);


Cartesian to Geographic - Simple Iteration
==========================================
```

```
Ellipsoid:
semi-major axis a =   6378137.000
flattening      f = 1/298.257222101
Cartesian coordinates:
X =  -3563081.362
Y =  -2057145.984
Z =  -4870449.482
Geodetic coordinates:
Latitude  =  -50  0  0.000000 (D M S)
Longitude = -150  0  0.000000 (D M S)
Height    =     10000.000
Iterations    =   3

>>
```

MATLAB function *Cart2Geo_ Newton.m*

```
function [lat,lon,h] = Cart2Geo_Newton(a,flat,X,Y,Z)
%
% [lat,lon,h] = Cart2Geo_Newton(a,flat,X,Y,Z)
%   Function computes the latitude (lat), longitude (lon) and height (h)
%   of a point related to an ellipsoid defined by semi-major axis (a)
%   and denominator of flattening (flat) given Cartesian coordinates
%   X,Y,Z.  Latitude and longitude are returned as radians.  This function
%   uses Newton-Raphson Iteration for converting X,Y,Z to lat,lon,height.

%------------------------------------------------------------------------
% Function:  Cart2Geo_Newton()
%
% Usage:     [lat,lon,h] = Cart2Geo_Newton(a,flat,X,Y,Z);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   3 March 2008
%
% Functions required:
%    radii()
%
% Purpose:
%    Function Cart2Geo_Newton() will compute latitude, longitude (both in
%    radians) and height of a  point related to an ellipsoid defined by
%    semi-major axis (a) and denominator of flattening (flat) given
%    Cartesian coordinates X,Y,Z.
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    c       - cosine(lat)
%    count   - integer counter for number of iterations
%    corrn   - correction to approximate value
%    d2r     - degree to radian conversion factor = 57.29577951...
%    e2      - 1st eccentricity squared
%    ep2     - 2nd eccentricity squared (ep = e-primed)
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    F       - function
%    dF      - derivative of function
%    h       - height above ellipsoid
%    lat     - latitude (radians)
```

```
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor-axis of ellipsoid
%    rm      - radius of curvature of meridian section of ellipsoid
%    rp      - radius of curvature of prime vertical section of ellipsoid
%    s       - sine(lat)
%
% Remarks:
%    This function uses Newton-Raphson Iteration, see Refences [1].
%
% References:
% [1] Deakin, R.E. and Hunter, M.N., 2008, GEOMETRIC GEODESY - PART A,
%       School of Mathematical and Geospatial Sciences, RMIT University,
%       Melbourne, AUSTRALIA, March 2008.

%--------------------------------------------------------------------------

% Set degree to radian conversion factor
d2r = 180/pi;

% calculate flattening f and ellipsoid constant e2
f   = 1/flat;
e2  = f*(2-f);
ep2 = e2/(1-e2);

% compute 1st approximation of geodetic latitude for Newton_Raphson
% Iteration
p   = sqrt(X*X + Y*Y);
lat = atan(Z/(p*(1-e2)));
corrn = 1;
count = 0;
while (abs(corrn)>1e-10)
    % Compute radii of curvature
    [rm,rp] = radii(a,flat,lat);
    s = sin(lat);
    c = cos(lat);
    %   Compute value of function and its derivative for approximate latitude
    F  = Z + rp*e2*s - p*s/c;
    dF = rm*ep2*c - p/c/c;
    corrn = F/dF;
    new_lat = lat - corrn;
    count = count+1;
    lat = new_lat;
end;

% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);

% compute longitude and height
lon = atan2(Y,X);
h   = p/cos(lat) - rp;

% Print results to screen
fprintf('\n\nCartesian to Geographic - Newton');
fprintf('\n================================');
fprintf('\nEllipsoid:');
fprintf('\nsemi-major axis a = %13.3f',a);
fprintf('\nflattening      f = 1/%13.9f',flat);
fprintf('\nCartesian coordinates:');
fprintf('\nX = %13.3f',X);
fprintf('\nY = %13.3f',Y);
fprintf('\nZ = %13.3f',Z);
fprintf('\nGeodetic coordinates:');
[D,M,S] = DMS(lat*d2r);
if D == 0 && lat < 0
    fprintf('\nLatitude  =  -0 %2d %9.6f (D M S)',M,S);
```

```
else
    fprintf('\nLatitude  = %4d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon*d2r);
if D == 0 && lon < 0
    fprintf('\nLongitude =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude = %4d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\nHeight    = %13.3f',h);
fprintf('\nIterations    = %3d',count);
fprintf('\n\n');
```

Help message for MATLAB function *Cart2Geo_Newton.m*

```
>> help Cart2Geo_Newton

  [lat,lon,h] = Cart2Geo_Newton(a,flat,X,Y,Z)
    Function computes the latitude (lat), longitude (lon) and height (h)
    of a point related to an ellipsoid defined by semi-major axis (a)
    and denominator of flattening (flat) given Cartesian coordinates
    X,Y,Z.  Latitude and longitude are returned as radians.  This function
    uses Newton-Raphson Iteration for converting X,Y,Z to lat,lon,height.

>>
```

Output from MATLAB function *Cart2Geo_Newton.m*

```
>> d2r = 180/pi;
>> a = 6378137;
>> flat = 298.257222101;
>> lat = -50/d2r;
>> lon = -150/d2r;
>> h = 10000;
>> [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
>> [X,Y,Z]'

ans =

         -3563081.36230554
         -2057145.98367164
         -4870449.48202417

>> [lat,lon,h] = Cart2Geo_Newton(a,flat,X,Y,Z);


Cartesian to Geographic - Newton
================================
Ellipsoid:
semi-major axis a =   6378137.000
flattening      f = 1/298.257222101
Cartesian coordinates:
X =  -3563081.362
Y =  -2057145.984
Z =  -4870449.482
Geodetic coordinates:
Latitude  =  -49 59 60.000000 (D M S)
Longitude = -150  0  0.000000 (D M S)
Height    =     10000.000
Iterations    =   2

>>
```

MATLAB function *Cart2Geo_ Bowring2.m*

```
function [lat,lon,h] = Cart2Geo_Bowring2(a,flat,X,Y,Z)
%
% [lat,lon,h] = Cart2Geo_Bowring2(a,flat,X,Y,Z)
%   Function computes the latitude (lat), longitude (lon) and height (h)
%   of a point related to an ellipsoid defined by semi-major axis (a)
%   and denominator of flattening (flat) given Cartesian coordinates
%   X,Y,Z.  Latitude and longitude are returned as radians.  This function
%   uses Bowring's method for converting X,Y,Z to lat,lon,height and
%   Newton-Raphson iteration.

%-----------------------------------------------------------------------
% Function:  Cart2Geo_Bowring2()
%
% Usage:     [lat,lon,h] = Cart2Geo_Bowring2(a,flat,X,Y,Z);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   14 February 2008
%            Version  1.1   11 June 2012
%
% Functions required:
%    radii()
%
% Purpose:
%    Function Cart2Geo_Bowring2() will compute latitude, longitude (both in
%    radians) and height of a  point related to an ellipsoid defined by
%    semi-major axis (a) and denominator of flattening (flat) given
%    Cartesian coordinates X,Y,Z.  This function uses Newton-Raphson
%    iteration.
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    b       - semi-minor axis of ellipsoid
%    c       - cos(psi)
%    c3      - cos(psi) cubed
%    d2r     - degree to radian conversion factor = 57.29577951...
%    e2      - 1st eccentricity squared
%    ep2     - 2nd eccentricity squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor-axis of ellipsoid
%    psi     - parametric latitude (radians)
%    rm      - radius of curvature of meridian section of ellipsoid
%    rp      - radius of curvature of prime vertical section of ellipsoid
%    s       - sin(psi)
%    s3      - sin(psi) cubed
%
% Remarks:
%    This function uses Bowring's method with Newton-Raphson iteration to
%    solve for the parametric latitude, see Ref [1].
%    Bowring's method is also explained in References [2] and [3].
%
% References:
% [1] Bowring, B.R., 1976, 'Transformation from spatial to
%       geographical coordinates', Survey Review, Vol. XXIII,
%       No. 181, pp. 323-327.
% [2] Gerdan, G.P. & Deakin, R.E., 1999, 'Transforming Cartesian
```

```
%       coordinates X,Y,Z to geogrpahical coordinates phi,lambda,h', The
%       Australian Surveyor, Vol. 44, No. 1, pp. 55-63, June 1999.
% [3] Deakin, R.E. and Hunter, M.N., 2013, GEOMETRIC GEODESY (Part A),
%       School of Mathematical and Geospatial Sciences, RMIT University,
%       Melbourne, AUSTRALIA, Jan 2013.

%--------------------------------------------------------------------------

% Set degree to radian conversion factor
d2r = 180/pi;

% calculate flattening f and ellipsoid constants e2, ep2 and b
f   = 1/flat;
e2  = f*(2-f);
ep2 = e2/(1-e2);
b   = a*(1-f);

% compute 1st approximation of parametric latitude psi
p   = sqrt(X*X + Y*Y);
psi = atan(Z/(p*(1-f)));

% compute parametric latitude from Bowring's equation by Newton-Raphson
% iteration
corrn = 1;
count = 0;
while (abs(corrn)>1e-10)
    s  = sin(psi);
    s3 = s*s*s;
    c  = cos(psi);
    c2 = c*c;
    c3 = c2*c;
    t  = tan(psi);
    %   Compute value of function and its derivative for approximate latitude
    F  = p*t - a*e2*c3*t - (1-f)*(Z + b*ep2*s3);
    dF = p/c2 - a*e2*c;
    corrn = F/dF;
    new_psi = psi - corrn;
    count = count+1;
    psi = new_psi;
    % If there are more thna five iterations then break out of while loop.
    if count>5
        break;
    end;
end;

% compute latitude
lat = atan(tan(psi)/(1-f));
% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);

% compute longitude and height
lon = atan2(Y,X);
h   = p/cos(lat) - rp;

% Print results to screen
fprintf('\n\nCartesian to Geographic - Bowring''s Iterative method');
fprintf('\n==================================================');
fprintf('\nEllipsoid:');
fprintf('\nsemi-major axis a = %13.3f',a);
fprintf('\nflattening      f = 1/%13.9f',flat);
fprintf('\nCartesian coordinates:');
fprintf('\nX = %13.3f',X);
fprintf('\nY = %13.3f',Y);
fprintf('\nZ = %13.3f',Z);
fprintf('\nGeodetic coordinates:');
```

```
[D,M,S] = DMS(lat*d2r);
if D == 0 && lat < 0
    fprintf('\nLatitude  =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  = %4d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon*d2r);
if D == 0 && lon < 0
    fprintf('\nLongitude =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude = %4d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\nHeight    = %13.3f',h);
fprintf('\nIterations    = %3d',count);
fprintf('\n\n');
```

Help message for MATLAB function *Cart2Geo_Bowring2.m*

```
>> help Cart2Geo_Bowring2
  [lat,lon,h] = Cart2Geo_Bowring2(a,flat,X,Y,Z)
    Function computes the latitude (lat), longitude (lon) and height (h)
    of a point related to an ellipsoid defined by semi-major axis (a)
    and denominator of flattening (flat) given Cartesian coordinates
    X,Y,Z.  Latitude and longitude are returned as radians.  This function
    uses Bowring's method for converting X,Y,Z to lat,lon,height and
    Newton-Raphson iteration.
```

Output from MATLAB function *Cart2Geo_Bowring2.m*

```
>> d2r = 180/pi;
>> a = 6378137;
>> flat = 298.257222101;
>> lat = -50/d2r;
>> lon = -150/d2r;
>> h = 10000;
>> [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
>> [X,Y,Z]'

ans =

         -3563081.36230554
         -2057145.98367164
         -4870449.48202417

>> [lat,lon,h] = Cart2Geo_Bowring2(a,flat,X,Y,Z);

Cartesian to Geographic - Bowring's Iterative method
=====================================================
Ellipsoid:
semi-major axis a =   6378137.000
flattening      f = 1/298.257222101
Cartesian coordinates:
X =  -3563081.362
Y =  -2057145.984
Z =  -4870449.482
Geodetic coordinates:
Latitude  =  -50  0  0.000000 (D M S)
Longitude = -150  0  0.000000 (D M S)
Height    =     10000.000
Iterations    =   2

>>
```

MATLAB function *Cart2Geo_ Lin.m*

```
function [lat,lon,h] = Cart2Geo_Lin(a,flat,X,Y,Z)
%
% [lat,lon,h] = Cart2Geo_Lin(a,flat,X,Y,Z)
%   Function computes the latitude (lat), longitude (lon) and height (h)
%   of a point related to an ellipsoid defined by semi-major axis (a)
%   and denominator of flattening (flat) given Cartesian coordinates
%   X,Y,Z.  Latitude and longitude are returned as radians.  This function
%   uses Lin & Wang's method for converting X,Y,Z to lat,lon,height.

%--------------------------------------------------------------------------
% Function:  Cart2Geo_Lin()
%
% Usage:     [lat,lon,h] = Cart2Geo_Lin(a,flat,X,Y,Z);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   3 March 2008
%
% Purpose:
%    Function Cart2Geo_Lin() will compute latitude, longitude (both in
%    radians) and height of a  point related to an ellipsoid defined by
%    semi-major axis (a) and denominator of flattening (flat) given
%    Cartesian coordinates X,Y,Z.
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    b       - semi-minor axis of ellipsoid
%    count   - integer counter for number of iterations
%    corrn   - correction to approximate value
%    d2r     - degree to radian conversion factor = 57.29577951...
%    e2      - 1st eccentricity squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    func    - function
%    funcp   - derivative of function
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor-axis of ellipsoid
%    pQ      - perpendicular distance of Q on ellipsoid from minor-axis
%    q       - multiplying factor
%    ZQ      - Z-coord of Q on ellipsoid
%
% Remarks:
%    This function uses Newton-Raphson Iteration, see Refences [1].
%    X,Y,Z are coords of P in space.  Q is the projection of P onto the
%    reference ellipsoid via the normal.  XQ,YQ,ZQ are the coords of Q on
%    the ellipsoid.
%
% References:
% [1] Lin, K.C. & Wang, J., 1995, 'Transformation from geocentric to
%       geodetic coordinates using Newton's iteration.', Bulletin
%       Geodesique, Vol. 69, pp. 300-303.
% [2] Deakin, R.E. and Hunter, M.N., 2008, GEOMETRIC GEODESY - PART A,
%       School of Mathematical and Geospatial Sciences, RMIT University,
%       Melbourne, AUSTRALIA, March 2008.

%--------------------------------------------------------------------------
```

```
% Set degree to radian conversion factor
d2r = 180/pi;

% calculate flattening f, semi-minor axis length b and e2
f  = 1/flat;
b  = a*(1-f);
e2 = f*(2-f);

% compute powers of a and b
a2   = a*a;
a4   = a2*a2;
b2   = b*b;
b4   = b2*b2;
ab   = a*b;
a2b2 = a2*b2;

% compute powers of X,Y,Z coords of P
X2 = X*X;
Y2 = Y*Y;
Z2 = Z*Z;
p2 = X2 + Y2;
p  = sqrt(p2);

% compute 1st approximation of multiplying factor q
A = a2*Z2 + b2*p2;
q = (ab*sqrt(A)*A - a2b2*A)/(2*(a4*Z2 + b4*p2));

% Newton-Raphson Iteration
% The test for convergence if when F approaches zero.
count = 0;
while 1
    %   Compute value of function and its derivative for approximate latitude
    twoq = 2*q;
    A1   = a2+twoq;
    A2   = A1*A1;
    A3   = A2*A1;
    B1   = b2+twoq;
    B2   = B1*B1;
    B3   = B2*B1;
    F    = a2*p2/A2 + b2*Z2/B2 - 1;
    % Test to see if F is sufficiently close to zero, and if so, then break
    % out of the while loop.
    if abs(F)<1e-12
        break;
    end;
    dF   = -4*(a2*p2/A3 + b2*Z2/B3);
    corrn = F/dF;
    new_q = q - corrn;
    count = count+1;
    q = new_q;
    % If there are more thna five iterations then break out of while loop.
    if count>5
        break;
    end;
end;
% compute Z- and p-coord of Q on the ellipsoid
twoq = 2*q;
pQ = a2*p/(a2+twoq);
ZQ = b2*Z/(b2+twoq);

% compute latitude, longitude and height
lat = atan(ZQ/(pQ*(1-e2)));
lon = atan2(Y,X);
h   = sqrt((p-pQ)^2+(Z-ZQ)^2);
```

```
if p+abs(Z)<pQ+abs(ZQ)
    h = -h;
end;

% Print results to screen
fprintf('\n\nCartesian to Geographic - Lin & Wang');
fprintf('\n====================================');
fprintf('\nEllipsoid:');
fprintf('\nsemi-major axis a = %13.3f',a);
fprintf('\nflattening      f = 1/%13.9f',flat);
fprintf('\nCartesian coordinates:');
fprintf('\nX = %13.3f',X);
fprintf('\nY = %13.3f',Y);
fprintf('\nZ = %13.3f',Z);
fprintf('\nGeodetic coordinates:');
[D,M,S] = DMS(lat*d2r);
if D == 0 && lat < 0
    fprintf('\nLatitude  =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  = %4d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon*d2r);
if D == 0 && lon < 0
    fprintf('\nLongitude =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude = %4d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\nHeight    = %13.3f',h);
fprintf('\nIterations    = %3d',count);
fprintf('\n\n');
```

Help message for MATLAB function *Cart2Geo_Lin.m*

```
>> help Cart2Geo_Lin

  [lat,lon,h] = Cart2Geo_Lin(a,flat,X,Y,Z)
    Function computes the latitude (lat), longitude (lon) and height (h)
    of a point related to an ellipsoid defined by semi-major axis (a)
    and denominator of flattening (flat) given Cartesian coordinates
    X,Y,Z.  Latitude and longitude are returned as radians.  This function
    uses Lin & Wang's method for converting X,Y,Z to lat,lon,height.

>>
```

Output from MATLAB function *Cart2Geo_Lin.m*

```
>> d2r = 180/pi;
>> a = 6378137;
>> flat = 298.257222101;
>> lat = -50/d2r;
>> lon = -150/d2r;
>> h = 10000;
>> [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
>> [X,Y,Z]'

ans =

         -3563081.36230554
         -2057145.98367164
         -4870449.48202417
```

```
>> [lat,lon,h] = Cart2Geo_Lin(a,flat,X,Y,Z);


Cartesian to Geographic - Lin & Wang
====================================
Ellipsoid:
semi-major axis a =   6378137.000
flattening      f = 1/298.257222101
Cartesian coordinates:
X =  -3563081.362
Y =  -2057145.984
Z =  -4870449.482
Geodetic coordinates:
Latitude  =  -49 59 60.000000 (D M S)
Longitude = -150  0  0.000000 (D M S)
Height    =     10000.000
Iterations    =   1

>>
```

MATLAB function *Cart2Geo_ Paul.m*

```
function [lat,lon,h] = Cart2Geo_Paul(a,flat,X,Y,Z)
%
% [lat,lon,h] = Cart2Geo_Paul(a,flat,X,Y,Z)
%   Function computes the latitude (lat), longitude (lon) and height (h)
%   of a point related to an ellipsoid defined by semi-major axis (a)
%   and denominator of flattening (flat) given Cartesian coordinates
%   X,Y,Z.  Latitude and longitude are returned as radians.  This function
%   uses Paul's direct method for converting X,Y,Z to lat,lon,height.

%---------------------------------------------------------------------------
% Function:  Cart2Geo_Paul()
%
% Usage:     [lat,lon,h] = Cart2Geo_Paul(a,flat,X,Y,Z);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   3 March 2008
%
% Purpose:
%    Function Cart2Geo_Paul() will compute latitude, longitude (both in
%    radians) and height of a  point related to an ellipsoid defined by
%    semi-major axis (a) and denominator of flattening (flat) given
%    Cartesian coordinates X,Y,Z.
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    a2      - a-squared
%    alpha   - variable in Paul's method
%    beta    - variable in Pauls method
%    d2r     - degree to radian conversion factor = 57.29577951...
%    e2      - 1st eccentricity squared
%    e4      - e2-squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor-axis of ellipsoid
```

```
%    p2      - p-squared
%    q       - numeric term in cubic equation in u
%    t       - single real-root of cubic equation in t
%    u       - single real-root of cubic equation in u
%    X,Y,Z   - Cartesian coordinates
%    X2,Y2,Z2  powers of X,Y,Z coords
%    zeta    - solution of quartic in terms of t
%
% Remarks:
%    This function uses Pauls' direct method, see Refences [1] & [2].
%
% References:
% [1] Paul, M.K., 1973, 'A note on computation of geodetic coordinates grom
%       geocentric (cartesian) coordinates', Bulletin Geodesique, No. 108,
%       pp. 134-139.
% [2] Deakin, R.E. and Hunter, M.N., 2008, GEOMETRIC GEODESY - PART A,
%       School of Mathematical and Geospatial Sciences, RMIT University,
%       Melbourne, AUSTRALIA, March 2008.

%-------------------------------------------------------------------------

% Set degree to radian conversion factor
d2r = 180/pi;

% calculate f, e2 and a-squared
f  = 1/flat;
e2 = f*(2-f);
e4 = e2*e2;
a2 = a*a;

% compute powers of X,Y,Z coords of P
X2 = X*X;
Y2 = Y*Y;
Z2 = Z*Z;
p2 = X2 + Y2;
p  = sqrt(p2);

% compute alpha, beta and squared values
alpha  = (p2+a2*e4)/(1-e2);   % ref [2], eqn (314), p.103
alpha2 = alpha*alpha;
beta   = (p2-a2*e4)/(1-e2);   % ref [2], eqn (310), p.102
beta2  = beta*beta;

% compute q
A  = beta+Z2;
q  = 1 + (27*Z2*(alpha2-beta2))/(2*A*A*A);  % ref [2], eqn (324), p.105
q2 = q*q;

% compute u
B = sqrt(q2-1);
u = 1/2*((q+B)^(1/3) + (q-B)^(1/3));  % ref [2], eqn (323), p. 104

% compute t
t = A/6*u + Z2/12 - beta/6;  % ref [2], eqn (321), p. 104

% compute zeta
root1 = sqrt(t);
if Z<0
    root1 = -root1;
end;
root2 = sqrt(Z2/4 - beta/2 - t + alpha*Z/4/root1);
if Z<0
    root2 = -root2;
end;
zeta = root1 + root2;
```

```
% compute latitude, longitude and height
lat = atan((zeta + Z/2)/p);

% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);

% compute longitude and height
lon = atan2(Y,X);
h   = p/cos(lat) - rp;

% Print results to screen
fprintf('\n\nCartesian to Geographic - Paul');
fprintf('\n==============================');
fprintf('\nEllipsoid:');
fprintf('\nsemi-major axis a = %13.3f',a);
fprintf('\nflattening      f = 1/%13.9f',flat);
fprintf('\nCartesian coordinates:');
fprintf('\nX = %13.3f',X);
fprintf('\nY = %13.3f',Y);
fprintf('\nZ = %13.3f',Z);
fprintf('\nGeodetic coordinates:');
[D,M,S] = DMS(lat*d2r);
if D == 0 && lat < 0
    fprintf('\nLatitude  =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  = %4d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon*d2r);
if D == 0 && lon < 0
    fprintf('\nLongitude =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude = %4d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\nHeight    = %13.3f',h);
fprintf('\n\n');
```

Help message for MATLAB function *Cart2Geo_Paul.m*

```
>> help Cart2Geo_Paul

  [lat,lon,h] = Cart2Geo_Paul(a,flat,X,Y,Z)
    Function computes the latitude (lat), longitude (lon) and height (h)
    of a point related to an ellipsoid defined by semi-major axis (a)
    and denominator of flattening (flat) given Cartesian coordinates
    X,Y,Z.  Latitude and longitude are returned as radians.  This function
    uses Paul's direct method for converting X,Y,Z to lat,lon,height.

>>
```

Output from MATLAB function *Cart2Geo_ Paul.m*

```
>> d2r = 180/pi;
>> a = 6378137;
>> flat = 298.257222101;
>> lat = -50/d2r;
>> lon = -150/d2r;
>> h = 10000;
>> [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
>> [X,Y,Z]'

ans =

         -3563081.36230554
         -2057145.98367164
         -4870449.48202417


>> [lat,lon,h] = Cart2Geo_Paul(a,flat,X,Y,Z);


Cartesian to Geographic - Paul
==============================
Ellipsoid:
semi-major axis a =   6378137.000
flattening      f = 1/298.257222101
Cartesian coordinates:
X =  -3563081.362
Y =  -2057145.984
Z =  -4870449.482
Geodetic coordinates:
Latitude  =  -50  0  0.000000 (D M S)
Longitude = -150  0  0.000000 (D M S)
Height    =     10000.000

>>
```

MATLAB function *radii.m*

MATLAB functions *Geo2Cart.m* *Cart2Geo_Substitution.m* *Cart2Geo_Newton.m* *Cart2Geo_Bowring.m* *Cart2Geo_Lin.m* and *Cart2Geo_Paul.m* call function *DMS.m* to convert decimal degrees to degrees, minutes and seconds (for printing). This function has been printed in a previous section. Also, all the functions, except *Cart2Geo_Lin.m* call function *radii.m* to compute ellipsoid radii of curvature. This function is shown below.

MATLAB function *radii.m*

```
function [rm,rp] = radii(a,flat,lat)
%
% [rm,rp]=radii(a,flat,lat)  Function computes radii of curvature in
%   the meridian and prime vertical planes (rm and rp respectively) at a
%   point whose latitude (lat) is known on an ellipsoid defined by
%   semi-major axis (a) and denominator of flattening (flat).
%   Latitude must be in radians.
%   Example: [rm,rp] = radii(6378137,298.257222101,-0.659895044);
%            should return rm = 6359422.96233327 metres and
%                          rp = 6386175.28947842 metres
%            at latitude -37 48 33.1234 (DMS) on the GRS80 ellipsoid

%----------------------------------------------------------------------------
% Function:  radii(a,flat,lat)
%
% Syntax:    [rm,rp] = radii(a,flat,lat);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  1 August 2003
%            Version  2.0  6 April  2006
%            Version  3.0  9 February 2008
%
% Purpose:   Function radii() will compute the radii of curvature in
%            the meridian and prime vertical planes, rm and rp respectively
%            for the point whose latitude (lat) is given for an ellipsoid
%            defined by its semi-major axis (a) and denominator of
%            flattening (flat).
%
% Return value: Function radii() returns rm and rp
%
% Variables:
%  a      - semi-major axis of spheroid
%  c      - polar radius of curvature
%  c1     - cosine of latitude
%  c2     - cosine of latitude squared
%  e2     - 1st-eccentricity squared
%  ep2    - 2nd-eccentricity squared (ep2 means e-prime-squared)
%  f      - flattening of ellipsoid
%  lat    - latitude of point (radians)
%  rm     - radius of curvature in the meridian plane
%  rp     - radius of curvature in the prime vertical plane
%  V      - latitude function defined by V-squared = sqrt(1 + ep2*c2)
%  V2,V3  - powers of V
```

```
%
% Remarks:
%  Formulae are given in [1] (section 1.3.9, page 85) and in
%  [2] (Chapter 2, p. 2-10) in a slightly different form.
%
% References:
% [1] Deakin, R.E. and Hunter, M.N., 2008, GEOMETRIC GEODESY – PART A,
%       School of Mathematical and Geospatial Sciences, RMIT University,
%       Melbourne, AUSTRALIA, March 2008.
% [2] THE GEOCENTRIC DATUM OF AUSTRALIA TECHNICAL MANUAL, Version 2.2,
%       Intergovernmental Committee on Surveying and Mapping (ICSM),
%       February 2002 (www.anzlic.org.au/icsm/gdatum)
%---------------------------------------------------------------------------

% compute flattening f, polar radius of curvature c and 2nd-eccentricity
% squared ep2
f   = 1/flat;
c   = a/(1-f);
e2  = f*(2-f);
ep2 = e2/(1-e2);

% calculate the square of the sine of the latitude
c1 = cos(lat);
c2 = c1*c1;

% compute latitude function V
V2 = 1+(ep2*c2);
V  = sqrt(V2);
V3 = V2*V;

% compute radii of curvature
rm = c/V3;
rp = c/V;
```

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